Variational approximation and finite elements

Exercise 1

Let $f \in C^0([0,1])$, $k \in C^1([0,1])$ such that $k(x) > 0$ for all $x \in [0,1]$ and $a > 0$. We consider the following problem

$$(P): \qquad -\big(k(x)u'(x)\big)' + au(x) = f(x), \text{ in }]0,1[, \qquad u(0) = 0 \quad \text{and} \quad k(1)u'(1) = 1.$$

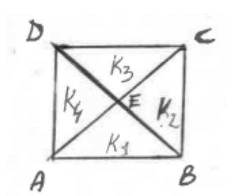
1. *Write the variational formulation associated to (P) using an appropriate Hilbert space V.*
2. *We now consider a uniform discretization $(x_i)_{i=0,\ldots,N}$ of the domain $[0,1]$, made of N intervals of length $h = \frac{1}{N}$. Let V_h be the subspace of V made of continous functions in $[0,1]$ such that the restriction to each interval $[x_i, x_{i+1}]$ is a polynomial function of degree 1 ($\mathbb{P}_1$ approximation).*
   (a) *Determinate the dimension of V_h and define the basis functions.*
   (b) *Write the discrete variational formulation.*
   (c) *Write the final linear system to solve, making explicit the different coefficients.*
3. *Explain how to treat a non homogeneous Dirichlet condition $u(0) = b$ ($b \in \mathbb{R}$).*

Exercise 2

Let Ω be the unit square $]0,1[\times]0,1[$. We consider the variational problem:

$$\text{Find } u \in H_0^1(\Omega) \text{ such that } \int_\Omega \Big(\partial_x u \partial_x v + \partial_y u \partial_y v\Big) dxdy = \int_\Omega v dxdy, \quad \forall v \in H_0^1(\Omega).$$

We divide Ω into four triangles as illustrated in the following figure:



We denote u_h the approximate solution of u obtained by a $\mathbb{P}^1$ finite element method using this triangularization. We emphasize that it is very simple discretization and that it cannot be used to solve an industrial problem.

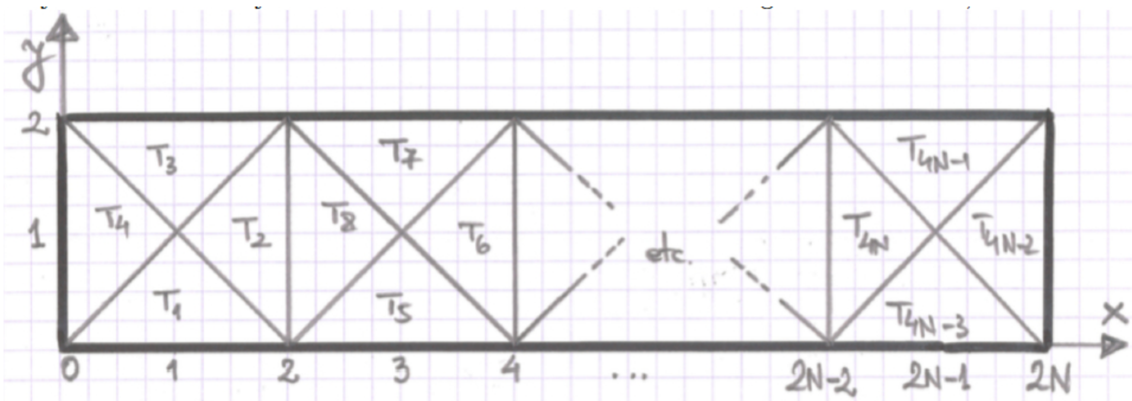
1. *Define the approximate space V_h and show that it is a space of dimension 1.*
2. *Let ω be the unique function in V_h that has a value equal to 1 in E. We write $u_h = U\omega$ with $U \in \mathbb{R}$. Determinate the coefficients a and b (depending on ω) such that the problem to solve is the linear equation $aU = b$.*
3. *Compute a and b. Deduce that $U = \frac{1}{12}$.*

Exercise 3

Let N be a positive integer and Ω the rectangle from $\mathbb{R}^2$ with vertices $(0,0)$, $(0,2)$, $(2N,0)$ and $(2N,2)$ (see figure below). We consider the variational problem: to find $u \in H_0^1(\Omega)$, such that

$$\forall v \in H_0^1(\Omega), \quad \int_\Omega \Big(\partial_x u \partial_x v + \partial_y u \partial_y v\Big) dxdy = \int_\Omega v dxdy.$$

We take for granted that this problem has a unique solution. We propose to approach numerically the solution u by the finite element method based on triangular $\mathbb{P}^1$ elements, on the mesh:



1. *Define the approximation space V_N. What is its dimension ?*
2. *Give explicitly a basis of V_N.*
3. *Find the explicit numerical solution of the approximated problem.*