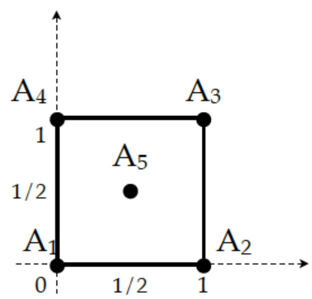
Unisolvent finite elements

Exercise 1
Let K be the square $K = [0,1] \times [0,1]$ with the following nodes: the four vertices and its center.



We consider the linear space $\mathcal{P} = \text{span}\left\{1, x, y, xy, y^2, x^2\right\}$ *and the set of linear forms on* $\mathcal{P}$ *given by:*

$$\Sigma = \left\{p \to p(A_1), p \to p(A_2), p \to p(A_3), p \to p(A_4), p \to p(A_5), p \to \int_K p(x,y)dxdy\right\}.$$

Is the finite element $\{K, \Sigma, \mathcal{P}\}$ *unisolvent? Depending on your answer, find a non zero element of* $\mathcal{P}$ *cancelling all forms of* Σ*, or find the corresponding basis functions.*

Exercise 2
Let K be a triangle with vertices a_1, a_2, a_3. We denotes a_{ij} (for $1 \leq i < j \leq 3$) the middle of the edges $[a_i, a_j]$ and we define by $a_{iij} = (2a_i + a_j)/3$ (for $1 \leq i \neq j \leq 3$) the six points placed on the edges in order to divide them into three uniform intervals. Let $S_1 = \{a_{ij}\}_{1\leq i<j\leq 3}$ and $S_2 = \{a_{iij}\}_{1\leq i\neq j\leq 3}$.
For $k = 1, 2$, is the set S_k $\mathbb{P}_k$-unisolvent ? If yes, express the basis functions of the finite element $\{K, \mathbb{P}_k, S_k\}$ using the barycentric coordinates.

Exercise 3
Let K be a triangle with vertices a_1, a_2, a_3. We denotes a_{ij} (for $1 \leq i < j \leq 3$) the middle of the edges and a_0 the triangle barycenter. We denotes λ_i, for $i = 1, 2, 3$, the barycentric coordinates and we consider the space

$$\mathcal{P} = \text{span}\left\{\lambda_1^2,\ \lambda_2^2,\ \lambda_3^2,\ \lambda_1\lambda_2,\ \lambda_1\lambda_3,\ \lambda_2\lambda_3,\ \lambda_1\lambda_2\lambda_3\right\}.$$

Finally, Σ is the set of linear functionals in $\mathcal{P}$ that assign to a_i, a_{ij} and a_0 (7 points in total) the values of the unknown function. Is the finite element $\{K, \Sigma, \mathcal{P}\}$ unisolvent ? Depending on your answer, find a non zero function in $\mathcal{P}$ that vanishes for all the elements in Σ or find the basis functions of this finite element.