Variational formulations and Lax-Milgram theorem for multi-dimensional problems

Exercise 1
Let Ω an open subset of $\mathbb{R}^2$ and $f \in L^2(\Omega)$. Find a variational formulation associated to the problem

$$\begin{cases} -\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y^2} = f, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

Prove the existence and uniqueness of a solution to this variational problem. Then, show that there exists a constant $C > 0$ such that, for all $f \in L^2(\Omega)$,

$$\|u\|_{H^1(\Omega)} \leq C\|f\|_{L^2(\Omega)}.$$

Exercise 2
Let Ω an open bounded and connected subset of $\mathbb{R}^d$ with a regular boundary. For $(u, v) \in H^1(\Omega) \times H^1(\Omega)$, we consider the bilinear form

$$a(u, v) = \int_\Omega \nabla u \nabla v dx + \left(\fint_\Omega u dx \right) \left(\fint_\Omega v dx \right),$$

where $\fint_\Omega v dx = \int_\Omega v dx / \int_\Omega dx$.

1. *Thanks to the Rellich theorem, prove the Poincaré-Wirtinger inequality which says that it exists a constant $C > 0$, such that*
$$\forall\, u \in H^1(\Omega), \qquad \|u - \left(\fint_\Omega u dx \right) \|_{L^2(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)}.$$
2. *Show that a is coercive.*
3. *Deduce that for all $f \in L^2(\Omega)$, it exists a unique solution $u \in H^1(\Omega)$ to the problem*
$$a(u, v) = \int_\Omega f v dx, \quad \forall v \in H^1(\Omega).$$
4. *Then, show that $\fint_\Omega u dx = \int_\Omega f dx$. Interpret which boundary value problem corresponds to this variational formulation, assuming its solution belongs to $H^2(\Omega)$.*

Exercise 3 (Exam '22)
Let Ω an open subset of $\mathbb{R}^2$ with a regular boundary $\partial\Omega, \lambda > 0, f \in L^2(\Omega)$ and $k \in L^\infty(\Omega)$ such that $k(\mathbf{x}) \geq k_{\min} > 0$ a.e. We consider the variational problem

$$\text{(FV)} : \textit{Find } u \in H^1(\Omega) \textit{ such that } \int_\Omega k \nabla u \cdot \nabla v d\mathbf{x} + \int_\Omega u v d\mathbf{x} + \lambda \int_{\partial\Omega} u v ds = \int_\Omega f v d\mathbf{x}, \quad \forall v \in H^1(\Omega).$$

1. *Show that the problem (FV) has a unique solution and then that there exists a constant $C > 0$ such that*
$$\|u\|_{H^1(\Omega)} \leq C\|f\|_{L^2(\Omega)}.$$

2. *Explain if the terms of the Green formula*

$$\int_\Omega \mathrm{div}\mathrm{w}\, v d\mathbf{x} = -\int_\Omega \mathbf{w}\cdot\nabla v d\mathbf{x} + \int_{\partial\Omega} \mathbf{w}\cdot\mathbf{n} v ds$$

are well defined in the following cases :

- $\mathbf{w} \in L^2(\Omega)^2$ *and* $v \in H^1(\Omega)$,
- $\mathbf{w} \in H^1(\Omega)^2$ *and* $v \in H^1(\Omega)$.

3. *Interpret the variational formulation (FV) in terms of a boundary value problem when its solution is in* $H^2(\Omega)$.

Exercise 4

Let Ω *an open bounded subset of* $\mathbb{R}^d$ *with a regulary boundary* Γ.

1. *Show that the application*

$$v \to N(v) = \left(\|\nabla v\|^2_{L^2(\Omega)} + \|\gamma_0(v)\|^2_{L^2(\Gamma)}\right)^{\frac{1}{2}} = \left(\int_\Omega |\nabla v|^2 dx + \int_\Gamma v^2 d\sigma\right)^{\frac{1}{2}}$$

is a norm in $H^1(\Omega)$ *equivalent to the usual norm.*

2. *Let* $f \in L^2(\Omega)$ *and* $g \in L^2(\Gamma)$. *Show that, for all* $\alpha > 0$, *the problem*

$$\text{Find } u \in H^1(\Omega), \qquad \int_\Omega \nabla u\cdot\nabla v dx + \alpha\int_\Gamma uv d\sigma = \int_\Omega fv dx + \int_\Gamma gv d\sigma, \quad \forall v \in H^1(\Omega),$$

has a unique solution. Interpret this formulation in terms of a boundary value problem when its solution is in $H^2(\Omega)$.

3. *Which boundary value problem is it for* $\alpha = 0$ *? Taking for instance* $\Omega =]0,1[$ *and* $g = 0$, *determinate an* $\alpha < 0$ *such that the problem has not a unique solution.*