Variational formulations and Lax-Milgram theorem for one dimensional problems

Exercise 1
Let $f \in L^2(0,1)$, we consider the following equation on $(0,1)$:

$$-u''(x) + u(x) = f(x).$$

For each set of boundary conditions below, write a variational formulation with its appropriate functional space, identify the bilinear an linear forms involved, and study the equivalence between variationnal and classical solutions :

1. $u(0) = u(1) = 0.$
2. $u(0) = 0, u(1) = 1.$
3. $u'(0) = 0, u'(1) = 1.$
4. $u'(0) + 2u(0) = 0, u(1) = 0.$

Exercise 2
We consider the problem

$$(P): \qquad -\big((1+x^2)u'\big)' + xu = f, \text{ in }]0,1[, \qquad u(0) = u(1) = 0,$$

where $f \in C^0([0,1])$.

1. What does it mean that u is a strong solution of (P) ?
2. Write a variational formulation (Q) associated to the problem (P).
3. Show that (Q) has a unique solution.
4. Does the solution of (Q) is also a strong solution of (P) ?
5. How will you treat the problem

$$(P_2): \qquad -\big((1+x^2)u'\big)' + xu = f, \text{ in }]0,1[, \qquad u(0) = u_0 \text{ and } u(1) = u_1,$$

   where u_0 and u_1 are two real numbers ?

Exercise 3
Solve the problem : Find $u \in H_0^1(0,1)$ such that

$$\int_0^1 u'(x)v'(x)dx = v\Big(\frac{1}{2}\Big), \quad \forall v \in H_0^1(0,1).$$

Is the solution in $H^2(0,1)$?

For $v \in H^1(0,1)$, we remind that $v(y) = v(x) + \int_x^y v'(t)dt$ for all $(x,y) \in [0,1]^2$. A consequence is that it exists $C > 0$ such that for all $v \in H^1(0,1)$, $\sup_{y\in[0,1]} |v(y)| \leq C\|v\|_{H^1(0,1)}$ (see Exercise 2 of previous exercise sheet).

Exercise 4

Let $f \in L^2(0,1)$ such that $\int_0^1 f(x)dx = 0$. We consider the problem

$$(P): \qquad -u'' = f, \quad \text{in }]0,1[, \qquad u'(0) = u'(1) = 0 \qquad \text{and} \qquad \int_0^1 u(x)dx = 0.$$

1. Discuss why we assume $\int_0^1 f(x)dx = 0$ and why we add the condition $\int_0^1 u(x)dx = 0$.
2. Write a variational formulation (Q) associated to (P), introducing the space

$$H^1_m(0,1) = \{v \in H^1(0,1), \int_0^1 v(x)dx = 0\}.$$

3. Show that (Q) has a unique solution. For that, you need to prove the Poincaré-Wirtinger inequality which says that it exists a constant $C > 0$, such that

$$\|u\|_{L^2} \leq C\|u'\|_{L^2}, \qquad \forall\, u \in H^1_m(0,1).$$

4. Prove that (P) admits a unique solution in $H^2(0,1)$.

Exercise 5

Let $p, q \in C^1([0,1])$, $r \in C^0([0,1])$ and $f \in L^2(0,1)$. We also assume that $r(x) \geq 0$ for all $x \in [0,1]$ and that there exists $\alpha > 0$ such that $p(x) \geq \alpha$ for all $x \in [0,1]$. We consider the problem

$$(P): \qquad -(pu')' + qu' + ru = f, \text{ in }]0,1[, \qquad u(0) = u(1) = 0.$$

1. Write a variational formulation (Q) associated to (P) introducing a bilinear form that is not symmetric.
2. Show for instance that if q is a decreasing function then the problem (Q) has a unique solution.

   Notice that instead of assuming that q is a decreasing function, we could for instance make an hypothesis such that $||q||_\infty \leq \alpha$. However, there exists some q for which the Lax-Milgram theorem cannot be applied (for instance $q(x) = \lambda x$ with $\lambda \in \mathbb{R}$ positive enough).
3. To avoid the hypothesis done in the previous question, we will write a variational formulation that has a symmetric bilinear form, introducing

$$h(x) = e^{-\int_0^x \frac{q(y)}{p(y)}dy}.$$

   (a) Show that (P) can be rewritten

$$-(hpu')' + hru = hf, \qquad u(0) = u(1) = 0.$$

   (b) Deduce another variational formulation $(Q2)$ associated to (P).
   (c) Show that $(Q2)$ has a unique solution.
   (d) Prove that (P) admits a unique solution in $H^2(0,1)$.

Exercise 6 (Exam '18)

Let $p \in \mathbb{R}$, and $f \in L^2(0,1)$. We consider the following $1D$ problem :

$$(S) \begin{cases} -u'' + u = f, & \text{on }]\,0,1[, \\ u(0) = u(1), \\ u'(0) = u'(1) + p. \end{cases}$$

We set $V = \big\{u \in H^1(0,1), u(0) = u(1)\big\}$.

1. Show that a $H^2(0,1)$ solution of (S) is solution of the following variational formulation :

$$(VF) \quad \text{Find } u \in V, \quad \forall v \in V, \quad \int_0^1 \big(u'(x)v'(x) + u(x)v(x)\big)\, dx = \int_0^1 f(x)v(x)dx - pv(0).$$

2. For $v \in H^1(0,1)$, we remind that $v(y) = v(x) + \int_x^y v'(t)dt$ for any $(x,y) \in [0,1]^2$. Show there exists $C > 0$ such that for all $v \in H^1(0,1), \sup_{y\in[0,1]} |v(y)| \leq C\|v\|_{H^1(0,1)}$.
3. Show existence and uniqueness of u verifying (VF).
4. Prove that this solution belongs to $H^2(0,1)$.
5. Deduce that (S) has a unique $H^2(0,1)$ solution.