Sobolev spaces

Exercise 1

1. Show that $H^1(\mathbb{R})$ is not a subset of $L^1(\mathbb{R})$.
2. Show that if Ω is an open bounded subset of $\mathbb{R}$, then

$$L^2(\Omega) \subset L^1(\Omega) \quad \text{and} \quad H^1(\Omega) \subset L^1(\Omega).$$

Exercise 2 We would like to prove that every function $v \in H^1(0,1)$ is continuous (meaning there exists a continuous function in its class), and that for all $x \in [0,1]$, the map $v \to v(x)$ is a linear continuous map from $H^1(0,1)$ to $\mathbb{R}$.

1. Let $w(x)$ be the function defined for $x \in [0,1]$ by

$$w(x) = \int_0^x v'(t)dt.$$

   - Show that w is well defined and continuous on $[0,1]$.
   - Show that $w' = v'$ in the sense of distributions.
   - Deduce that there exists a continuous function in the class of v (that we still denote v) and that for all $(x,y) \in [0,1]^2$,

$$v(y) = v(x) + \int_x^y v'(t)dt.$$

2. Show that for all $x \in [0,1]$, $v \to v(x)$ is a linear continuous form on $H^1(0,1)$.

Exercise 3 Let B the open ball of $\mathbb{R}^d$ of radius a, with $0 < a < 1$.

1. If $d = 2$, show that the function $u(\mathbf{x}) = |\ln \|\mathbf{x}\|_2|^\alpha$ belongs to $H^1(B)$ for $0 < \alpha < \frac{1}{2}$, but is unbounded in a neighborhood of 0.
2. If $d \geq 3$, show that the function defined by $u(\mathbf{x}) = (\|\mathbf{x}\|_2)^{-\beta}$ belongs to $H^1(B)$ for $0 < \beta < (d-2)/2$, but is unbounded in a neighborhood of 0.

Theorem 1 (Poincaré Inequality) *Let Ω be an open subset of $\mathbb{R}^d$, bounded in at least one space direction (i.e. enclosed between two hyperplanes). There exists a constant $C > 0$ depending only on Ω such that, for all $v \in H_0^1(\Omega)$,*

$$\|v\|_{L^2(\Omega)} \leq C \, \|\nabla v\|_{L^2(\Omega)}.$$

Exercise 4

1. Show that Poincaré inequality does not hold in $H^1(\Omega)$.
2. Show that Poincaré inequality does not hold if $\Omega = \mathbb{R}^d$.

Theorem 2 (Trace theorem) *Let Ω be an open bounded subset of $\mathbb{R}^d$, whose boundary is lipschitz continuous. The trace map γ_0 defined by*

$$\gamma_0 : \begin{cases} \mathcal{D}(\overline{\Omega}) & \to & \mathcal{C}^0(\partial\Omega) \\ v & \to & \gamma_0(v) = v_{|\partial\Omega} \end{cases}$$

can be continuously extended into a linear continuous map from $H^1(\Omega)$ to $L^2(\partial\Omega)$ (still denoted γ_0). In particular, there exists $C > 0$ such that, $\forall v \in H^1(\Omega)$,

$$\|\gamma_0(v)\|_{L^2(\partial\Omega)} \leq C\|v\|_{H^1(\Omega)}.$$

Exercise 5 We aim to show that the application γ_0 is not continuous from $L^2(\Omega) \to L^2(\partial\Omega)$, that is, there does not exist $C > 0$ such that $\forall v \in L^2(\Omega)$,

$$\|v_{|\partial\Omega}\|_{L^2(\partial\Omega)} \leq C\|v\|_{L^2(\Omega)}.$$

We consider the case where $\Omega = B_1$. Build a sequence of regular functions in $\overline{\Omega}$ identically 1 on $\partial\Omega$ whose $L^2(\Omega)$ norm converges toward 0.

Exercise 6 The aim of that exercise is to show that the trace theorem does not hold when Ω is not regular enough. Let $\Omega \subset \mathbb{R}^2$ defined by $\Omega = \{(x, y) \in \mathbb{R}^2 \,; 0 < x < 1,\ 0 < y < x^r\}$, where $r > 2$. Let the function $v(x, y) = x^\alpha$. Show that $v \in H^1(\Omega)$ if and only if $2\alpha + r > 1$, while $v \in L^2(\partial\Omega)$ if and only if $2\alpha > -1$. Conclude.

Exercise 7 Let Ω be a bounded open subset of $\mathbb{R}^n$, with a regular boundary Γ. We set for $v \in H^2(\Omega)$,

$$|v|_2 = \left(\int_\Omega \sum_{|\alpha|=2} \frac{2}{\alpha!}|\partial^\alpha v|^2\right)^{\frac{1}{2}}$$

where $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| = \sum_{i=1}^n \alpha_i$, $\alpha! = \prod_{i=1}^n \alpha_i!$ and $\partial^\alpha v = \dfrac{\partial^{|\alpha|} v}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_n} x_n}$.

The space $H_0^2(\Omega)$ stands for the closure of $\mathcal{D}(\Omega)$ in $H^2(\Omega)$ for the norm $\|u\|_{H^2}^2 = \|u\|_{H^1}^2 + |u|_2^2$, and corresponds to elements of $H^2(\Omega)$ for which u and $\partial_n u$ (normal derivative) have a null trace on Γ.

1. Show that $\forall v \in H_0^2(\Omega), \qquad \|\triangle u\|_{L^2} = |u|_2$.
2. Starting from the usual Green formula, show that

$$\forall u \in H^4(\Omega), \quad \forall v \in H^2(\Omega), \qquad \int_\Omega \triangle u \triangle v dx = \int_\Omega (\triangle^2 u) v dx - \int_\Gamma (\partial_n \triangle u) v d\sigma + \int_\Gamma (\triangle u)\partial_n v d\sigma.$$