Refresher in distributions theory : Hints

Exercise 1 (Partial derivatives and Green formula) Let $u(x,y) = \ln\left(\frac{1}{\sqrt{x^2+y^2}}\right)$ and $r = \sqrt{x^2+y^2}$. We denote by B_r the open ball centered in 0 with radius r and $\Omega_\epsilon = B_1 \setminus \bar{B}_\epsilon$. We aim at computing $\triangle u$ in $\mathcal{D}'(B_1)$.

1. Compute $\triangle u(x,y)$ for $(x,y) \neq 0$.
   We get $\partial_x u = -x/r^2$ and $\partial_{xx} u = (x^2-y^2)/r^4$. Similarly $\partial_{yy} u = (y^2-x^2)/r^4$, so that $\Delta u = 0$.
2. For $\varphi \in \mathcal{D}(B_1)$, we set

$$I_\varepsilon = \int_{\Omega_\varepsilon} u \triangle\varphi \, \mathrm{d}x\mathrm{d}y.$$

   Compute I_ϵ.
   Using twice Green's formula, we get

$$I_\epsilon = -\int_{\partial\Omega_\epsilon} \nabla u \cdot \mathbf{n}\, \varphi \, \mathrm{d}\sigma + \int_{\partial\Omega_\epsilon} u\, \nabla\varphi \cdot \mathbf{n} \, \mathrm{d}\sigma.$$

   Using the fact that $\partial\Omega_\epsilon = S(0,1) \cup S(0,\epsilon)$ and that, on $S(0,\epsilon)$, the unit exterior normal writes $\mathbf{n} = -\mathbf{x}/\|\mathbf{x}\| = -\mathbf{x}/r$, with $\mathbf{x} = (x,y)^T$, it leads to

$$I_\epsilon = \underbrace{-\frac{1}{\epsilon}\int_{S(0,\epsilon)} \varphi \, \mathrm{d}\sigma}_{I_\epsilon^1} + \underbrace{\frac{\ln(\epsilon)}{\epsilon}\int_{S(0,\epsilon)} \nabla\varphi \cdot \mathbf{x} \, \mathrm{d}\sigma}_{I_\epsilon^2}.$$

3. Compute $\lim_{\epsilon\to 0} I_\epsilon$. Deduce the expression of $\triangle u$ in $\mathcal{D}'(B_1)$.
   We prove that $I_\epsilon^2 \to 0$ as $\epsilon \to 0$, and using the mean-value theorem we show that $I_\epsilon^1 \to -2\pi\varphi(0,0)$ as $\epsilon \to 0$ (to obtain this limit, you could also use polar coordinates and the dominated convergence theorem). So that $I_\epsilon \to -2\pi\varphi(0,0)$ as $\epsilon \to 0$. Consequently, $\Delta u = -2\pi\delta_{(0,0)}$ in $\mathcal{D}'(B_1)$.

Exercise 2 (Heat Kernel)

1. Let $f \in L^1(\mathbb{R})$, and set, for $\varepsilon > 0$, $f_\varepsilon(x) = \varepsilon^{-1} f(x/\varepsilon)$. Determine $\lim_{\varepsilon \to 0} f_\varepsilon$ in the sense of distributions.
   Using Lebesgue's dominated convergence theorem and making the change of variable $y = \frac{x}{\varepsilon}$, we prove that, for any $\varphi \in \mathcal{D}(\mathbb{R})$,
   $$\langle T_{f_\epsilon}, \varphi \rangle \to \left(\int_{\mathbb{R}} f(y) \mathrm{d}y \right) \langle \delta_0, \varphi \rangle \text{ as } \epsilon \to 0.$$
2. Let $u(t,x) = \frac{1}{(4\pi t)^{1/2}} \exp\left(\frac{-x^2}{4t}\right)$. Show that u solves the heat equation $\partial_t u(t,x) = \partial_{xx} u(t,x)$, for $t \in]0, +\infty[$, $x \in \mathbb{R}$.
   Direct computations give $\partial_t u(t,x) = \left(\frac{x^2}{4t^2} - \frac{1}{2t}\right) u(t,x)$ for $t > 0$ and $x \in \mathbb{R}$. Same result for $\partial_{xx} u(t,x)$.
3. Determine the distributional limits : $\lim_{t \to 0} u(t,x)$ and $\lim_{t \to 0} \partial_t u(t,x)$. Deduce the PDE satisfied by v in $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R})$, where $v(t,x) = u(t,x) H(t)$.
   Question 1) using $f(y) = \frac{1}{\pi^{1/2}} \exp(-y^2)$ gives that $u(t,.) \underset{t \to 0}{\to} \delta_0$ in $\mathcal{D}'(\mathbb{R})$ and using Question 2) we prove that $\partial_t u(t,.) \underset{t \to 0}{\to} \Delta \delta_0$ in $\mathcal{D}'(\mathbb{R})$. We get
   $$\partial v - \Delta v = \delta_{(t,x)=(0,0)} \text{ in } \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}).$$