Refresher in distributions theory

Reminder : A distribution T on an open subset Ω of $\mathbb{R}^d$ is a linear continuous form on $\mathcal{D}(\Omega)$. This means that $T \in \mathcal{D}'(\Omega)$ if it is a linear application from $\mathcal{D}(\Omega)$ to $\mathbb{R}$ such that one of the two following equivalent properties holds : (i) If $(\varphi_n)_{n\in\mathbb{N}} \subset \mathcal{D}(\Omega)$ is such that there exists a compact subset K of Ω with $\operatorname{supp}\varphi_n \subset K$ for all n, and φ_n and all its (partial) derivatives converges uniformly to 0 on K, then $\langle T, \varphi_n\rangle \to 0$ as $n \to \infty$. (ii) For all compact subset K of Ω, there exists C_K, α_K such that for all $\varphi \in \mathcal{D}(\Omega)$ with support included in K,

$$|\langle T, \varphi\rangle| \leq C_K \sup_{|\alpha|\leq\alpha_K} \left\|\partial^{(\alpha)}\varphi\right\|_\infty .$$

Exercise 1 Are the following applications from $\mathcal{D}(\mathbb{R}^n)(n = 1 \text{ or } 2)$ to $\mathbb{R}$ distributions ?

a. $T : \varphi \mapsto |\varphi(0)|$,

b. $T : \varphi \mapsto \int_{\mathbb{R}^2} xy\varphi''(\sqrt{x^2+y^2})\mathrm{d}x\mathrm{d}y$,

c. $T : \varphi \mapsto \int_{\mathbb{R}} |x|^\alpha \varphi(x)\mathrm{d}x$, with $\alpha \in \mathbb{R}$ such that $\alpha > -1$,

d. $T : \varphi \mapsto \int_{\mathbb{R}} \ln|x|\varphi(x)\mathrm{d}x$.

Exercise 2

- Show that the application $\delta_0 : \varphi \in \mathcal{D}(\mathbb{R}) \mapsto \varphi(0)$, defines a distribution. We call it the Dirac mass at 0.
- Compute the derivatives of δ_0.

Now, our aim is to show that the Dirac distribution δ_0 is not regular, i.e., that there is no $g \in L^1_{loc}(\mathbb{R})$ such that :

$$\int_{\mathbb{R}} g(x)\varphi(x)\mathrm{d}x = \varphi(0), \quad \forall\varphi \in \mathcal{D}(\mathbb{R}).$$

Let us consider the sequence of functions $(\varphi_n(x))_n$ defined by :

$$\varphi_n(x) = \begin{cases} e^{1-\frac{1}{1-(nx)^2}} & \text{if } |x| < \frac{1}{n}, \\ 0 & \text{otherwise.} \end{cases}$$

- Show that $\varphi_n \in \mathcal{D}(\mathbb{R})$.
- Deduce the claimed result.

Exercise 3 (First order differential equation) Let $T \in \mathcal{D}'(\mathbb{R})$ and $a \in \mathbb{R}$. The aim of this exercise is to solve the differential equation

$$T' - aT = \delta$$

where δ is the Dirac distribution.

1. Prove that $T' = 0$ if and only if $T = K$ where K is a regular distribution associated to a (almost everywhere) constant function.
   For that, we will use that if $\varphi_0 \in \mathcal{D}(\mathbb{R})$ such that $\int_{\mathbb{R}} \varphi_0(x)\mathrm{d}x = 1$ then
   $$\forall\, \varphi \in \mathcal{D}(\mathbb{R}),\ \exists!\ \psi \in \mathcal{D}(\mathbb{R}), \exists!\ c \in \mathbb{R},\ \varphi = \psi' + c\varphi_0.$$
2. Solve the differential equation $T' - aT = 0$.
3. Solve the differential equation $T' - aT = \delta$.

Exercise 4 (Partial derivatives and Green formula) Let $u(x,y) = \ln\left(\frac{1}{\sqrt{x^2+y^2}}\right)$ and $r = \sqrt{x^2+y^2}$. We denote by B_r the open ball centered in 0 with radius r and $\Omega_\epsilon = B_1 \setminus \bar{B}_\epsilon$. We aim at computing $\triangle u$ in $\mathcal{D}'(B_1)$.

1. Compute $\triangle u(x,y)$ for $(x,y) \neq 0$.
2. For $\varphi \in \mathcal{D}(B_1)$, we set
   $$I_\varepsilon = \int_{\Omega_\varepsilon} u \triangle\varphi \,\mathrm{d}x\mathrm{d}y.$$
   Compute I_ϵ.
3. Compute $\lim_{\epsilon\to 0} I_\epsilon$. Deduce the expression of $\triangle u$ in $\mathcal{D}'(B_1)$.

Exercise 5 (Heat Kernel)

1. Let $f \in L^1(\mathbb{R})$, and set, for $\varepsilon > 0$, $f_\varepsilon(x) = \varepsilon^{-1} f(x/\varepsilon)$. Determine $\lim_{\varepsilon\to 0} f_\varepsilon$ in the sense of distributions.
2. Let $u(t,x) = \frac{1}{(4\pi t)^{1/2}} \exp\left(\frac{-x^2}{4t}\right)$. Show that u solves the heat equation $\partial_t u(t,x) = \partial_{xx} u(t,x)$, for $t \in]0,+\infty[$, $x \in \mathbb{R}$.
3. Determine the distributional limits : $\lim_{t\to 0} u(t,x)$ and $\lim_{t\to 0} \partial_t u(t,x)$. Deduce the PDE satisfied by v in $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R})$, where $v(t,x) = u(t,x)H(t)$.