Tug of War games and PDEs on graphs with applications in image and high dimensional data processing

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The game

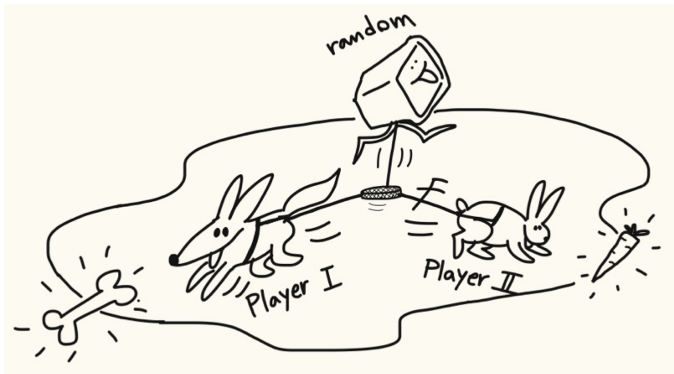


Figure 1. Illustration of the the game from [3].

The Tug-of-war game (TOW):

- Two player random turn zero-sum game played on a domain $\Omega \subset \mathbb{R}^N$ with a running payoff function $h : \Omega \to \mathbb{R}$ and a payoff function g defined on $\partial\Omega$.
- Setting:
  - A token is placed at an initial position $x_0 \in \Omega$ and each player can move the token to a new position in $B_\epsilon(x_k)$.
  - A fair coin is tossed: if player I wins, the token is moved to the position x_k^{I}, otherwise to the position x_k^{II}.
  - The game stops when the token reaches some position $x_f \in \partial\Omega$. In that case, the game stops and player I's payoff is $g(x_f) + \epsilon^2 \sum_{i=1}^{f-1} h(x_i)$.
  - The Dynamic programming principle reduces to

$$\begin{cases} u_\epsilon(x) = \frac{1}{2}\Big\{ \sup\limits_{y\in B_\epsilon(x)} u_\epsilon(y) + \inf\limits_{y\in B_\epsilon(x)} u_\epsilon(y) \Big\} + \epsilon^2 h(x) & \text{in } \Omega, \\ u_\epsilon(x) = g(x) & \text{on } \partial\Omega. \end{cases} \tag{1}$$

  - A General version: each player chooses a position with probabilities α and β, respectively, and that the game position moves with a uniform probability γ, with $\frac{\alpha+\beta}{2} + \gamma = 1$. In this case, the DPP reads

$$u_\epsilon(x) = \frac{\alpha}{2} \sup_{y\in B_\epsilon(x)} u_\epsilon(y) + \frac{\beta}{2} \inf_{y\in B_\epsilon(x)} u_\epsilon(y) + \gamma \fint_{B_\epsilon(x)} u_\epsilon(y)\mathrm{d}y + \epsilon^2 h(x), \quad x \in \Omega. \tag{2}$$

Connection with PDEs on graphs

Differential operators

Given a weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, w)$. The discrete upwind/donwind gradients of u are defined by

$$\nabla_w^\pm u(x) = (\partial_y^\pm u(x))_{y\in\mathcal{V}}^T, \text{ where } \partial_y^\pm u(x) = \left(\sqrt{w(x,y)}(u(y) - u(x))\right)^\pm. \tag{3}$$

Its $\mathcal{L}_p$ norm is defined as

$$\|\nabla_w^\pm u(x)\|_p = \begin{cases} \max_{y\in I(x)} \left(\sqrt{w(x,y)}(u(y) - u(x))^\pm\right) & \text{for } p = \infty \\ \|(\nabla_w^\pm u)(x)\|_p = \left[\sum\limits_{y\in I(x)} w(x,y)^{p/2}\big(u(y) - u(x)\big)^\pm\right]^{\frac{1}{p}} & \text{for } 1 \le p < \infty. \end{cases} \tag{4}$$

- The 2-Laplacian on graph is defined by

$$(\Delta_{w,2}u)(x) = \frac{\sum_{y\in I(x)} w(x,y)u(y)}{\sum_{y\in I(x)} w(x,y)} - u(x). \tag{5}$$

- The ∞-Laplacian on graph is defined by

$$(\Delta_{w,\infty}u)(x) = \frac{1}{2}\Big(\|\nabla_w^+ u(x)\|_\infty - \|\nabla_w^- u(x)\|_\infty\Big). \tag{6}$$

- For $2 \le p < \infty$, the game p-Laplacian on graph is defined by

$$(\Delta_{w,p}^G u)(x) = \frac{p-2}{p}\Delta_{w,\infty}u(x) + \frac{2}{p}\Delta_{w,2}u(x). \tag{7}$$

Averaging operators

Let us consider an Euclidean graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, w)$ with $\mathcal{V} = \Omega \subset \mathbb{R}^N$ and a weight function

$$w(x,y) = \begin{cases} 1 \text{ if } |x - y| \le \epsilon, \\ 0 \text{ otherwise}. \end{cases} \tag{8}$$

Then, using (4), we easily get

$$\begin{aligned} \max_{y\in I(x)} u(y) &= \|\nabla_w^+ u(x)\|_\infty + u(x), \\ \min_{y\in I(x)} u(y) &= u(x) - \|\nabla_w^- u(x)\|_\infty. \end{aligned} \tag{9}$$

Plugging (9) in (2), we obtain the following interpretation of the generalized TOW game (2) in terms of a PDE:

$$-\Delta_{\alpha,\beta,\gamma}u(x) = h(x), \tag{10}$$

where $\Delta_{\alpha,\beta,\gamma}u(x) = \frac{\alpha}{2}\|\nabla_w^+ u(x)\|_\infty - \frac{\beta}{2}\|\nabla_w^- u(x)\|_\infty + \gamma\Delta_{w,2}u(x)$.
On a general weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, w)$, recall the following notations for nonlocal dilation, nonlocal erosion and nonlocal mean, respectively:

$$\begin{aligned} \mathrm{NLD}(u)(x) = \|\nabla_w^+ u(x)\|_\infty + u(x) &= u(x) + \max_{y\in I(x)} \left(\sqrt{w(x,y)}(u(y) - u(x))^+\right), \\ \mathrm{NLE}(u)(x) = u(x) - \|\nabla_w^- u(x)\|_\infty &= u(x) - \max_{y\in I(x)} \left(\sqrt{w(x,y)}(u(y) - u(x))^-\right), \\ \mathrm{NLM}(u)(x) = u(x) + \Delta_{w,2}u(x) &= \frac{\sum_{y\in I(x)} w(x,y)u(y)}{\sum_{y\in I(x)} w(x,y)}, \end{aligned} \tag{11}$$

Then, defining the following nonlocal averaging operator:

$$\mathrm{NLA}(u) := \frac{\alpha}{2}\mathrm{NLD}(u) + \frac{\beta}{2}\mathrm{NLE}(u) + \gamma\mathrm{NLM}(u), \tag{12}$$

equation (10) can be rewritten as

$$u(x) - \mathrm{NLA}(u)(x) = h(x). \tag{13}$$

To discretize such nonlocal PDEs on general weighted graphs, as it suffices in practice to implement the nonlocal mathematical morphology operators (11).

Recovered PDEs

- Eikonal equation ($\alpha = \gamma = 0$ and $\beta = 1$) : $\|\nabla_w^- u(x)\|_\infty = h(x)$.
- ∞-Laplacian ($\alpha = \beta = 1$ and $\gamma = 0$) : $-\Delta_{w,\infty}u(x) = h(x)$.
- Laplace equation ($\alpha = \beta = 0$ and $\gamma = 1$) : $-\Delta_{w,2}u(x) = h(x)$.
- Game p-Laplace equation ($\alpha = \beta = \frac{p-2}{p}$ and $\gamma = \frac{2}{p}$) : $-\Delta_{w,p}^G u(x) = h(x)$.

Simple algorithms for inverse problems

Consider a subset $A \subset \mathcal{V}$ consisting of vertices with the missing information.Then, many problems we encounter in image processing and computer vision can be recast in the form of interpolation problems, i.e., one seeks constructing new values starting from known values, which amounts to solve the Dirichlet problem

$$\begin{cases} -\Delta_{\alpha,\beta,\gamma}u = 0 & \text{in } A, \\ u = g & \text{on } \partial A. \end{cases} \tag{14}$$

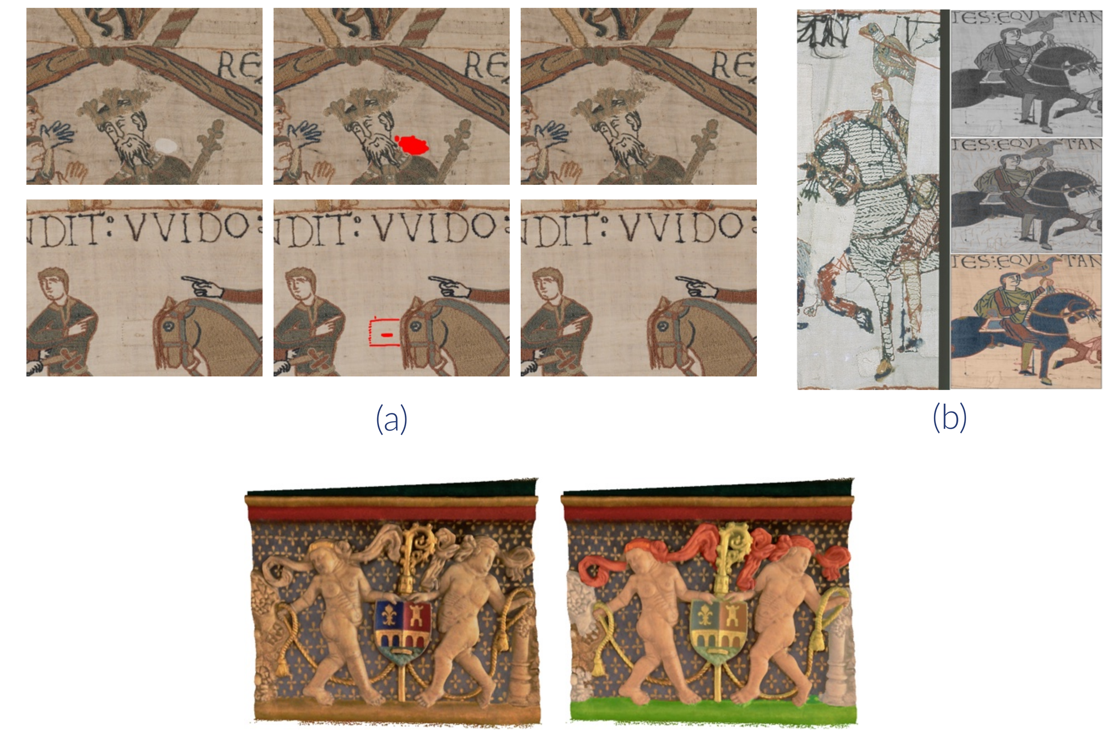
To solve (14) we consider the associated evolution problem, use a Euler discretization by taking $\partial_t u \approx \frac{u_{n+1}-u_n}{\Delta t}$, where $u_n(x) = u(x, n\Delta t)$. Taking $\Delta t = 1$ and $\Delta_{\alpha,\beta,\gamma} = \mathrm{NLA}(u) - u$, we get the following iterations governed by a nonlocal average filter consisting of convex combination of the nonlocal, dilation, erosion and mean terms:

$$\begin{cases} u_0 = u^0 & \text{in } A, \\ u_{n+1} = \mathrm{NLA}(u_n) & \text{in } A, \\ u_{n+1} = g & \text{on } \partial A. \end{cases} \tag{15}$$

Applications in cultural heritage

Nonlocal inpainting and 2D-3D colorization

We consider $A \subset \mathcal{V}$ the set of vertices with missing data and prescribe boundary condition on ∂A via a function $g : \partial A \to \mathbb{R}^3$. We apply the iterative scheme (15) with $\alpha = \beta = 1, \gamma = 0$, i.e., using the ∞-Laplacian.



(a) (b)

(c)

Figure 2. (a) virtual restoration of a 2D image of King Edward taken from the Bayeux Tapestry (original images, images with areas to be restored indicated in red, and restoration results). (b) 2D colorization, where the left column represents the backside of a scene from the Bayeux Tapestry, and the second column represents the initial image, the image with seeds, and the colored result. (c) 3D colorization, with input models on the left and colored ones on the right.

Semi-supervised segmentation

We apply the iterative scheme (15) with $\alpha = \gamma = 0, \beta = 1$, i.e., using the eikonal equation with $h \equiv 1$.

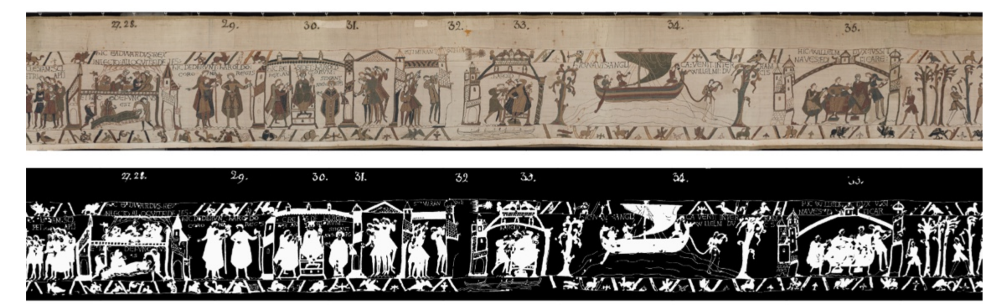


Figure 3. Semi-supervised segmentation of the Bayeux Tapestry.

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