Nonlinear Analysis 238 (2024) 113415

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

From weak to strong convergence of the gradients for Finsler p-Laplacian problems as $p \to \infty$

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ARTICLE INFO

Article history:
Received 20 May 2023
Accepted 6 October 2023
Communicated by Enrico Valdinoci

MSC:
35J92
35B40
28A20

Keywords:
Finsler p-Laplacian
Finsler metric
Convergence of the gradients

ABSTRACT

In this paper we investigate limits as $p \to \infty$ of solutions u_p to Finsler p-Laplacian problems $-\text{div}\left(F^*(x, \nabla u_p)^{p-1}\partial_\xi F^*(x, \nabla u_p)\right) = f$ with $f > 0$, coupled with a Dirichlet boundary condition $u_p = g$ on $\partial\Omega$. We prove that the whole sequence of solutions $\{u_p\}$ converges to the limit function u_∞ *strongly* in $W^{1,m}(\Omega)$ for any $1 \le m < \infty$, provided that $F^*(x, .)$ has some strict convexity on its unit sphere. We also characterize an explicit expression of the limit function.

1. Introduction

This paper is concerned with limits as $p \to \infty$ for Finsler p-Laplacian equations coupled with a Dirichlet boundary condition

$$\begin{cases} -\text{div}\left(F^*(x, \nabla u_p)^{p-1}\partial_\xi F^*(x, \nabla u_p)\right) = f & \text{in } \Omega \\ u_p = g & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where F^* is a Finsler metric, f is a positive continuous function on $\overline{\Omega}$ and $\partial_\xi F^*(x, .)$ is the subdifferential with respect to the second variable, which will be recalled in Section 2. It is well-known that these are the corresponding Euler–Lagrange equations of the following convex variational problems

$$\min_{u \in W^{1,p}(\Omega)} \left\{ \int_\Omega \frac{F^*(x, \nabla u)^p}{p} \mathrm{d}x - \int_\Omega uf\mathrm{d}x : \ u = g \text{ on } \partial\Omega \right\}.$$

When $F^*(x, \xi) = |\xi|$ is the Euclidean norm (independent of x), the Eqs. (1.1) become the standard p-Laplacian problems. The study of limits of p-Laplacian type problems as $p \to \infty$ has recently received a lot of attention and is closely connected with many relevant topics such as the Monge–Kantorovich transportation

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https://doi.org/10.1016/j.na.2023.113415

problem (see, for instance [2,18,26,29]), the sandpile problem [3,5,17], the mass optimization problem [9], the absolutely minimizing Lipschitz extensions, the infinity-harmonic functions and the Tug of War games (see, for instance [4,8,11,12,20] and the references therein).

As in Bhattacharya–DiBenedetto–Manfredi [7], when $f > 0$, $F^*(x,\xi) = |\xi|$ and $g = 0$, the sequence of solutions u_p is shown to be uniformly convergent to $u_\infty = d(x, \partial\Omega)$ (see also [22]). More recently, Buccheri–Leonori–Rossi [10] proposed two proofs showing the convergence of the gradients ∇u_p to the gradient of limit function ∇u_∞ strongly in $L^m(\Omega)$, $1 \le m < \infty$ as $p \to \infty$. The two proofs need special properties of the Euclidean norm: the first proof follows ideas given in [7] using the intrinsic characterization of norms induced by inner products called "*parallelogram law*", while the second one exploits an explicit expression of the subdifferential in terms of ξ, that is $\partial|\xi| = \frac{\xi}{|\xi|}$.

Unfortunately, as for the general case of Finsler metric F^*, the main difficulty is the lack of such a parallelogram law as well as an explicit expression of the subdifferential $\partial_\xi F^*(x,\xi)$.

As far as general Finsler metrics F^* are concerned, it is known that u_p converges uniformly to u_∞ on $\overline{\Omega}$ as $p \to \infty$ and the sequence of gradients $\{\nabla u_p\}_p$ is uniformly bounded in $L^m(\Omega)$ for any $1 \le m < \infty$ for large p (by a constant independent from p). Therefore, one can obtain the weak convergence of the gradients ∇u_p to ∇u_∞ in $L^m(\Omega)$ as $p \to \infty$ (see e.g. Section 3 below, also [16,21]). However, up to our knowledge, the strong convergence of ∇u_p as $p \to \infty$ is still missing in the general Finslerian setting.

The main goal of the present paper is to derive the strong convergence of ∇u_p in $L^m(\Omega)$, for any $1 \le m < \infty$, in the general Finslerian setting. This study confirms that the strong convergence of gradients ∇u_p still holds true, provided that $f > 0$ on Ω and F^* satisfies some geometric condition. More precisely, we will assume that $F^*(x,.)$ is strictly convex on its unit sphere for almost-everywhere x in Ω (see Section 2 for precise statements and examples). Under these conditions, we will show that the limit function u_∞ is actually the unique viscosity solution to the stationary Hamilton–Jacobi equation of eikonal type $F^*(x, \nabla u_\infty(x)) = 1$, coupled with the Dirichlet boundary condition $u_\infty = g$ on $\partial\Omega$. Moreover, the whole sequence $\{\nabla u_p\}$ is shown to be convergent to ∇u_∞ almost-everywhere in Ω as $p \to \infty$. After that, by making use of Clarkson's inequality and the reverse Fatou lemma, we obtain that ∇u_p converges to ∇u_∞ strongly in the Lebesgue spaces $L^m(\Omega)$ as $p \to \infty$ for any $1 \le m < \infty$. In other words, the whole sequence $\{u_p\}$ converges to the limit function u_∞ *strongly* in the Sobolev spaces $W^{1,m}(\Omega)$ for any $1 \le m < \infty$.

The paper is organized as follows. In the next section, we will present some preliminaries on Finsler metrics and assumptions. The weak convergence of the gradients ∇u_p is discussed in detail in Section 3. Section 4 is devoted to the limit variational problem and an expression of the limit function u_∞ when the source function f is positive. We also interpret the limit function in terms of viscosity solutions to an appropriate Hamilton–Jacobi equation of first order. Finally, in Section 5, we prove the strong convergence of the gradients ∇u_p to ∇u_∞ in $L^m(\Omega)$ as $p \to \infty$ for any $1 \le m < \infty$, and therefore the whole sequence $\{u_p\}$ converges to the limit function u_∞ strongly in the Sobolev spaces $W^{1,m}(\Omega)$ for any $1 \le m < \infty$.

2. Preliminaries and assumption

Let Ω be an open, connected, bounded Lipschitz domain of $\mathbb{R}^N$. A Finsler metric is a continuous function $F : \overline{\Omega} \times \mathbb{R}^N \longrightarrow [0, +\infty)$ such that $F(x,.)$ is sub-additive and positively 1-homogeneous with respect to the second variable, that is,

- $F(x, \xi_1 + \xi_2) \le F(x, \xi_1) + F(x, \xi_2) \ \ \forall x \in \overline{\Omega}, \xi_1, \xi_2 \in \mathbb{R}^N$;
- $F(x, t\xi) = tF(x,\xi) \ \ \forall (x,\xi) \in \overline{\Omega} \times \mathbb{R}^N$ and $t \ge 0$.

It is not difficult to see that the above two properties imply that $F(x,.)$ is convex.

In this work, the Finsler metric F is assumed to be non-degenerate, that is, there exist two positive constants K_1, K_2 such that

$$K_1|\xi| \leq F(x,\xi) \leq K_2|\xi| \text{ for all } x \in \overline{\Omega}, \xi \in \mathbb{R}^N. \tag{2.2}$$

Its polar function F^*, which is also a non-degenerate Finsler metric, is defined by

$$F^*(x,q) = \sup_{\{\xi:\, F(x,\xi)\leq 1\}} \langle q,\xi\rangle = \sup_{\{\xi:\, F(x,\xi)=1\}} \langle q,\xi\rangle = \sup_{\xi\neq 0} \frac{\langle q,\xi\rangle}{F(x,\xi)}.$$

The subdifferential with respect to the second variable of $F^*(x,.)$ is denoted by $\partial_\xi F^*(x,.)$, by definition,

$$\zeta \in \partial_\xi F^*(x,\xi_1) \Leftrightarrow F^*(x,\xi_2) \geq F^*(x,\xi_1) + \langle \zeta, \xi_2 - \xi_1\rangle \quad \forall \xi_2 \in \mathbb{R}^N.$$

The Finsler distance associated with F, denoted by d_F, is given by

$$d_F(x,y) = \inf_\eta \int_0^1 F(\eta(s),\dot{\eta}(s))\mathrm{d}s,$$

where the infimum is taken over all Lipschitz curves η joining x to y within $\overline{\Omega}$ and $\eta(0) = x, \eta(1) = y$.

Concerning the Dirichlet boundary condition, the function g needs to have some compatibility, that is $g(y) - g(x) \leq d_F(x,y)$ for all x,y on $\partial\Omega$. By extension if necessary, we assume moreover that $g \in W^{1,\infty}(\Omega)$ and $g(y) - g(x) \leq d_F(x,y)$ for all x,y in $\overline{\Omega}$.

In order to improve from the weak to strong convergence of the gradients, we will make use of the following assumption concerning the geometry of Finsler metric F^* (and, of course, F).

(A): $F^*(x,.)$ is strictly convex on its unit sphere in the sense that for any q_1, q_2 satisfying $F^*(x,q_1) = 1, F^*(x,q_2) = 1$ then

$$q_1 \neq q_2, t \in (0,1) \Rightarrow F^*(x,(1-t)q_1 + tq_2) < 1.$$

It is worth noting that the aforementioned property is related to the strict convexity of Banach spaces. As a sufficient condition, the assumption (A) holds if $F(x,.) \in C^1(\mathbb{R}^N \setminus \{0\})$, the set of continuously differentiable functions in $\mathbb{R}^N \setminus \{0\}$ (see e.g. [6, page 184]). As a typical example, the assumption (A) happens when $F(x,\xi) = k(x)\|\xi\|_p = k(x)\left(\xi_1^p + \cdots + \xi_N^p\right)^{\frac{1}{p}}$ for $1 < p < \infty$ and any positive continuous function k on $\overline{\Omega}$. In this case, we can check at once that $F^*(x,v) = \frac{1}{k(x)}\|v\|_{p'}$ with the Hölder conjugate $p' = \frac{p}{p-1}$.

3. Limit as $p \to \infty$: weak convergence

For $p > N$, we consider the following variational problem

$$\min_{u\in W^{1,p}(\Omega)} \left\{ \mathcal{F}_p(u) := \int_\Omega \frac{F^*(x,\nabla u)^p}{p}\mathrm{d}x - \int_\Omega uf\mathrm{d}x :\ u = g \text{ on } \partial\Omega \right\}.$$

Observe that the constraint set $W^{1,p}_g(\Omega) := \{u \in W^{1,p}(\Omega) : u = g \text{ on } \partial\Omega\}$ is a closed, convex, non-empty subset of the Sobolev space $W^{1,p}(\Omega)$. The objective functional $\mathcal{F}_p$ is coercive, strictly convex and lower semicontinuous on $W^{1,p}_g(\Omega)$. Therefore, following the standard variational arguments, the cost functional $\mathcal{F}_p$ admits a unique minimizer in $W^{1,p}_g(\Omega)$, which satisfies the Finsler p-Laplacian equation (1.1). Moreover, the sequence of solutions $\{u_p\}$ is known to be bounded in the Sobolev spaces $W^{1,m}(\Omega)$ for any $1 \leq m < \infty$, independently from p. So, one can obtain the weak convergence of the gradients ∇u_p to ∇u_∞ in $L^m(\Omega)$ as $p \to \infty$ (see, for example, [16,21]). Here we present the weak convergence for completeness.

Proposition 3.1. *Let u_p be a minimizer of $\mathcal{F}_p$ on $W^{1,p}_g(\Omega)$. Then, up to a subsequence, $u_p \rightrightarrows u_\infty$ uniformly on $\overline{\Omega}$ and $\nabla u_p \rightharpoonup \nabla u_\infty$ weakly in $L^m(\Omega)$ for any $1 \leq m < \infty$ as $p \to \infty$. Moreover one has $F^*(x,\nabla u_\infty(x)) \leq 1$ a.e. in Ω.*

Proof. Thanks to Theorem 2.E in [27], there is a Morrey-type inequality independent of $p > N+1$, that is,

$$\|u\|_{L^\infty(\Omega)} \le C_\Omega \|\nabla u\|_{L^p(\Omega)} \text{ for all } u \in W_0^{1,p}(\Omega), p > N+1, \tag{3.3}$$

where the constant C_Ω does not depend on p and u. Apply (3.3) to the function $u_p - g \in W_0^{1,p}(\Omega)$, we get

$$\|u_p - g\|_{L^\infty(\Omega)} \le C_\Omega \|\nabla u_p - \nabla g\|_{L^p(\Omega)}$$

and hence

$$\|u_p\|_{L^\infty(\Omega)} \le C_\Omega \|\nabla u_p\|_{L^p(\Omega)} + C_1. \tag{3.4}$$

On the other hand, since u_p is a minimizer and $g \in W_g^{1,p}(\Omega)$, $F^*(x, \nabla g(x)) \le 1$, we have

$$\begin{aligned}
\int_\Omega \frac{F^*(x,\nabla u_p)^p}{p} \mathrm{d}x &\le \int_\Omega u_p f \mathrm{d}x + \frac{|\Omega|}{p} - \int_\Omega g f \mathrm{d}x \\
&\le \|u_p\|_{L^\infty(\Omega)} \|f\|_{L^1(\Omega)} + \frac{|\Omega|}{p} - \int_\Omega g f \mathrm{d}x \\
&\le C_2 \|\nabla u_p\|_{L^p(\Omega)} + C_3 \quad (\text{by (3.4)}),
\end{aligned}$$

where the constants C_i do not depend on p. It follows from the non-degeneracy (2.2) of the Finsler metric F that

$$\|F^*(x, \nabla u_p)\|_{L^p(\Omega)}^p \le C_4 p (1 + \|F^*(x, \nabla u_p)\|_{L^p(\Omega)}).$$

It follows that

$$\|F^*(x, \nabla u_p)\|_{L^p(\Omega)} \le (C_5 p)^{\frac{1}{p-1}}. \tag{3.5}$$

Let $N < m \le p$. By the non-degeneracy again and the Morrey-Sobolev embedding theorem (see [1]), (3.5) yields

$$|u_p(x) - u_p(y)| \le C_6 |x-y|^{1-N/m}.$$

Thanks to Ascoli–Arzela's theorem, up to a subsequence, $u_p \rightrightarrows u_\infty$ uniformly on $\overline{\Omega}$ as $p \to \infty$ and

$$|u_\infty(x) - u_\infty(y)| \le C_6 |x-y|^{1-N/m}.$$

Letting $m \to \infty$, we deduce that $u_\infty \in W^{1,\infty}(\Omega)$.

We are now in a position to show that $F^*(x, \nabla u_\infty(x)) \le 1$ a.e. in Ω. Observe that $u_p \rightharpoonup u_\infty$ weakly in $W^{1,m}(\Omega)$, that is, $u_p \to u_\infty$ strongly in $L^m(\Omega)$ and $\nabla u_p \rightharpoonup \nabla u_\infty$ weakly in $L^m(\Omega)$ as $p \to \infty$. Now, fix any $\xi \in C_c(\Omega)$, $\xi \ge 0$. It follows from the convexity of $F^*(x, .)$, the weak convergence of $\xi \nabla u_p$, Hölder's inequality and (3.5) that

$$\begin{aligned}
\int_\Omega F^*(x, \nabla u_\infty) \xi \mathrm{d}x &= \int_\Omega F^*(x, \xi \nabla u_\infty) \mathrm{d}x \\
&\le \liminf_{p \to +\infty} \int_\Omega F^*(x, \xi \nabla u_p) \mathrm{d}x \\
&= \liminf_{p \to +\infty} \int_\Omega F^*(x, \nabla u_p) \xi \mathrm{d}x \\
&\le \liminf_{p \to +\infty} \left(\|\xi\|_{L^{p'}(\Omega)} \|F^*(x, \nabla u_p)\|_{L^p(\Omega)} \right) \\
&\le \liminf_{p \to +\infty} \left(|\Omega|^{\frac{m'-p'}{m'p'}} \|\xi\|_{L^{m'}(\Omega)} \|F^*(x, \nabla u_p)\|_{L^p(\Omega)} \right) \\
&\le |\Omega|^{\frac{m'-1}{m'}} \|\xi\|_{L^{m'}(\Omega)},
\end{aligned}$$

where $m' = \frac{m}{m-1}$, $p' = \frac{p}{p-1}$ are the Hölder conjugates of m and p. This implies that

$$\|F^*(x, \nabla u_\infty)\|_{L^m(\Omega)} \le |\Omega|^{\frac{m'-1}{m'}}.$$

Finally, the proof is completed by letting $m \to \infty$. □

4. The limit problem

4.1. The limit variational problem

In this subsection we characterize the limit problem as well as some properties of its solution, which will be useful later. Let us consider the following variational problem

$$\max_{u\in W^{1,\infty}(\Omega)}\left\{\int_\Omega uf\mathrm{d}x:\ F^*(x,\nabla u(x))\leq 1 \text{ a.e. in } \Omega,\ \ u=g \text{ on } \partial\Omega\right\}. \tag{4.6}$$

Proposition 4.2. *The limit function u_∞ is an optimal solution to the limit problem (4.6).*

Proof. Following Proposition 3.1, we get that $F^*(x,\nabla u_\infty(x))\leq 1$ a.e. in Ω as well as $u_\infty=g$ on $\partial\Omega$. In other words, the limit function u_∞ satisfies the constraints of (4.6). Now for any $v\in W^{1,\infty}_g(\Omega)$ satisfying $F^*(x,\nabla v(x))\leq 1$ a.e. in Ω, we get

$$-\int_\Omega u_p f\mathrm{d}x\leq \mathcal{F}_p(u_p)\leq \mathcal{F}_p(v)\leq \frac{|\Omega|}{p}-\int_\Omega vf\mathrm{d}x.$$

Letting $p\to\infty$ to get $\int_\Omega u_\infty f\mathrm{d}x\geq \int_\Omega vf\mathrm{d}x$, which shows the optimality of the limit function u_∞, and the proof is complete. □

Proposition 4.3. *Assume that $f>0$. Then the limit problem (4.6) has a unique optimal solution given by $\overline{u}(x)=\min_{y\in\partial\Omega}\{d_F(y,x)+g(y)\}$. In particular $u_\infty=\overline{u}$.*

A function u is called 1-Lipschitz with respect to d_F on $\overline{\Omega}$ (also, $1-d_F$ Lipschitz) if $u(y)-u(x)\leq d_F(x,y)$ for all x,y in $\overline{\Omega}$. Such a $1-d_F$ Lipschitz function u is completely characterized via their gradient by the condition $F^*(x,\nabla u(x))\leq 1$ a.e. in Ω. We give here a proof for completeness.

Lemma 4.4. *A function u is 1-Lipschitz with respect to d_F if and only if*

$$F^*(x,\nabla u(x))\leq 1 \ \ \text{a.e. in}\ \ \Omega.$$

Proof. We divide the proof into two parts. First, assume that u is 1-Lipschitz with respect to d_F. Then u is differentiable a.e. in Ω. Let $x\in\Omega$ be any point at which u is differentiable. We have, for any $v\in\mathbb{R}^N$,

$$\begin{aligned}\frac{\langle\nabla u(x),v\rangle}{F(x,v)}&=\lim_{h\to 0}\frac{u(x+hv)-u(x)}{F(x,hv)}\\&\leq\limsup_{h\to 0}\frac{d_F(x,x+hv)}{F(x,hv)}\\&\leq\limsup_{h\to 0}\frac{\int_0^1F(x+thv,hv)\mathrm{d}t}{F(x,hv)}=1.\end{aligned}$$

Hence, by definition, $F^*(x,\nabla u(x))\leq 1$. Conversely, suppose that $F^*(x,\nabla u(x))\leq 1$ a.e. in Ω. **Case 1:** If u is smooth then $F^*(x,\nabla u(x))\leq 1\ \forall x\in\overline{\Omega}$. For any x,y in $\overline{\Omega}$ and any Lipschitz curve η in $\overline{\Omega}$ joining x to y, we have

$$\begin{aligned}u(y)-u(x)&=\int_0^1\nabla u(\eta(t))\dot\eta(t)\mathrm{d}t\\&\leq\int_0^1F^*(\eta(t),\nabla u(\eta(t)))F(\eta(t),\dot\eta(t))\mathrm{d}t\\&\leq\int_0^1F(\eta(t),\dot\eta(t))\mathrm{d}t.\end{aligned}$$

It follows that $u(y) - u(x) \le d_F(x, y)$ for all x, y in $\overline{\Omega}$. **Case 2:** In general case, for any continuous function u satisfying $F^*(x, \nabla u(x)) \le 1$ a.e. $x \in \Omega$, there exists a sequence of compactly supported smooth functions u_ε such that $F^*(x, \nabla u_\varepsilon(x)) \le 1$ $\forall x \in \overline{\Omega}$ and $u_\varepsilon \rightrightarrows u$ uniformly on $\overline{\Omega}$ (see [25, Lemma 3.1]). According to Case 1 above, $u_\varepsilon(y) - u_\varepsilon(x) \le d_F(x, y)$ for all x, y in $\overline{\Omega}$. Passing to the limit we recover the same property for u, which completes the proof. □

Proof of Proposition 4.3. It is not difficult to see that $\overline{u}$ is 1-Lipschitz with respect to the Finsler distance d_F. Indeed, for any $x, z \in \overline{\Omega}$, there exists $y \in \partial\Omega$ such that $\overline{u}(x) = d_F(y, x) + g(y)$. Then

$$\overline{u}(z) - \overline{u}(x) \le d_F(y, z) + g(y) - d_F(y, x) - g(y) \le d_F(x, z).$$

This implies that $F^*(x, \nabla\overline{u}(x)) \le 1$ a.e. in Ω by Lemma 4.4. On the other hand, it is clear that $\overline{u}(x) \le g(x)$ for all x on $\partial\Omega$. Let us show the converse. Since the compatibility $g(x) - g(y) \le d_F(y, x)$, we get $g(x) \le d_F(y, x) + g(y)$ for all $y \in \partial\Omega$ and therefore $g \le \overline{u}$ on $\partial\Omega$. So $\overline{u}$ is admissible for the problem (4.6). We are now in a position to show that $\overline{u}$ is optimal and unique. Let $\overline{v}$ be an optimal solution. We will show that $\overline{v} = \overline{u}$. For any $x \in \overline{\Omega}$, $y \in \partial\Omega$, we have that

$$\overline{v}(x) - g(y) = \overline{v}(x) - \overline{v}(y) \le d_F(y, x).$$

Hence, $\overline{v}(x) \le d_F(y, x) + g(y)$ for all $y \in \partial\Omega$ and therefore $\overline{v}(x) \le \overline{u}(x)$ for all $x \in \overline{\Omega}$. Suppose, contrary to our claim, that $\overline{v}(x_0) < \overline{u}(x_0)$ for some $x_0 \in \overline{\Omega}$. By the continuity of f, $\overline{v}, \overline{u}$ and $f > 0$ in Ω, there exists a neighborhood U of x_0 in $\overline{\Omega}$ such that $\overline{v}(x) < \overline{u}(x)$ and $f(x) > 0$ for all x in U. It follows that $\int_\Omega \overline{v} f < \int_\Omega \overline{u} f$, which is a contradiction with the optimality of $\overline{v}$. As a consequence, we obtain that the limit function $u_\infty \equiv \overline{u} = \min_{y \in \partial\Omega} d_F(y, x) + g(y)$. □

4.2. Interpretation in terms of viscosity solutions

Let us recall briefly the notion of viscosity solutions and its construction via Perron's method. The viscosity solution concept was introduced in the early 1980s by Crandall and Lions [15] as a generalization of the classical solution to PDEs. Subsequently the definition and properties of viscosity solutions for Hamilton–Jacobi equations were refined in a joint work by Crandall, Evans and Lions [13]. In addition, an excellent reference for second-order equations might be *the user's guide* written by Crandall, Ishii, and Lions [14].

Let us consider a continuous Hamiltonian $H : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}$ satisfying the following assumptions, for any $x \in \overline{\Omega}$, $Z(x) := \{\xi \in \mathbb{R}^N : \; H(x, \xi) \le 0\}$:

(H1) coercivity: $Z(x)$ is compact;
(H2) quasiconvexity: $Z(x)$ is convex;
(H3) $H(x, 0) < 0$, that is $0 \in \operatorname{int} Z(x)$.

We are interested in the following quasiconvex Hamilton–Jacobi equation of first order

$$H(x, \nabla u) = 0, \;\; x \in \Omega. \tag{4.7}$$

For clarity, we recall below two equivalent definitions of viscosity solutions via touching functions and generalized differentials, following the presentation of a recent monograph [28].

Definition 4.5 (*Viscosity Solutions via Touching Functions*)**.** • A continuous function $u : \Omega \to \mathbb{R}$ is said to be a viscosity subsolution of (4.7) if $H(x, \nabla\phi(x)) \le 0$ for any $x \in \Omega$ and any C^1 function ϕ such that $u - \phi$ has a strict local maximum at x.

• A continuous function $u : \Omega \to \mathbb{R}$ is said to be a viscosity supersolution of (4.7) if $H(x, \nabla\phi(x)) \ge 0$ for any $x \in \Omega$ and any C^1 function ϕ such that $u - \phi$ has a strict local minimum at x.

• A function u is a viscosity solution of (4.7) if it is simultaneously a viscosity subsolution and a viscosity supersolution.

It is known that the viscosity solution concept can be also defined via generalized differentials. For any $x \in \Omega$, the sets

$$D^- u(x) = \left\{ p \in \mathbb{R}^N : \liminf_{y \to x} \frac{u(y) - u(x) - p(y-x)}{|y-x|} \geq 0 \right\}$$

$$D^+ u(x) = \left\{ p \in \mathbb{R}^N : \limsup_{y \to x} \frac{u(y) - u(x) - p(y-x)}{|y-x|} \leq 0 \right\}$$

are called the *(Frechét) subdifferential* and *superdifferential* of u at x, respectively.

Definition 4.6 (*An Equivalent Definition of Viscosity Solutions via Generalized Differentials*)**.** • A continuous function u is a viscosity subsolution of (4.7) if $H(x,p) \leq 0$ for every $x \in \Omega, p \in D^+u(x)$.
• A continuous function u is a viscosity supersolution of (4.7) if $H(x,p) \geq 0$ for every $x \in \Omega, p \in D^-u(x)$.
• We say that u is a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution.

For $x \in \overline{\Omega}$, let us define the support function of the 0-sublevel set $Z(x)$ by

$$\sigma(x,q) := \sup q \cdot Z(x) = \sup\{q \cdot \xi \mid \ \xi \in Z(x)\} \quad \text{for } q \in \mathbb{R}^N. \tag{4.8}$$

The assumptions (H1)–(H3) ensure that σ is a non-degenerate Finsler metric. Denote by $\Gamma(x,y)$ the set of all Lipschitz continuous curves ζ defined on $[0,1]$ joining x to y in $\overline{\Omega}$ and $\zeta(0) = x, \zeta(1) = y$. One then defines the intrinsic distance by

$$d_\sigma(x,y) := \inf_{\zeta \in \Gamma(x,y)} \int_0^1 \sigma(\zeta(t), \dot{\zeta}(t)) \mathrm{d}t,$$

which turns out to be a distance, but not necessarily symmetric. That is, $d_\sigma(x,y) \geq 0$ for any $x, y \in \overline{\Omega}$; $d_\sigma(x,y) = 0$ if and only if $y = x$. In addition, for all $x,y,z \in \overline{\Omega}$ one has $d_\sigma(x,y) \leq d_\sigma(x,z) + d_\sigma(z,y)$. We summarize below a basic character of viscosity subsolutions in terms of the intrinsic distance d_σ.

Proposition 4.7 (*[19,23]*)**.**

- *u is a viscosity subsolution if and only if $u(y) - u(x) \leq d_\sigma(x,y)$ for any $x, y \in \Omega$.*
- *Given g satisfying the compatibility condition $g(y) - g(x) \leq d_\sigma(x,y)$ for x, y in $\overline{\Omega}$, then the unique viscosity solution u^* to the Hamilton–Jacobi equation $H(x, \nabla u(x)) = 0$ in Ω, coupled with a Dirichlet boundary condition $u = g$ on $\partial\Omega$, is expressed by*

$$u^* = \max\left\{u : \ u \textit{ is a viscosity subsolution and } u = g \textit{ on } \partial\Omega\right\}.$$

Coming back to our concrete situation, let us set $H(x,\xi) := F^*(x,\xi) - 1$. Then the Hamiltionian H satisfies all the assumptions (H1)–(H3). Moreover, the support function σ defined in (4.8) is exactly the initial Finsler metric F.

Proposition 4.8. *The limit function u_∞ is a viscosity solution of the Hamilton–Jacobi equation $F^*(x, \nabla u) = 1$ and therefore, in particular*

$$F^*(x, \nabla u_\infty(x)) = 1 \textit{ a.e. in } \Omega.$$

Proof. Since g satisfies the compatibility condition $g(y) - g(x) \leq d_F(x,y)$ for all x, y in $\overline{\Omega}$, the unique viscosity solution u^* to the stationary Hamilton–Jacobi equation $H(x, \nabla u(x)) = F^*(x, \nabla u(x)) - 1 = 0$ in

Ω and $u = g$ on $\partial\Omega$, is given by

$$\begin{aligned} u^* &= \max\{u: \ u \text{ is a viscosity subsolution and } u = g \text{ on } \partial\Omega\} \\ &= \max\{u: \ u(y) - u(x) \le d_F(x,y) \text{ for all } x, y \text{ in } \Omega \text{ and } u = g \text{ on } \partial\Omega\} \\ &= \text{argmax}\,(4.6) = u_\infty. \quad \Box \end{aligned}$$

5. Limit as $p \to \infty$: strong convergence

In this section we will show an improvement on the weak in order to obtain the strong convergence of the gradients ∇u_p in the general Finslerian setting.

Theorem 5.9. *Assume that $F^*(x, .)$ satisfies the condition (A) about the strict convexity on its unit sphere for a.e. x in Ω and $f > 0$ in Ω. Let u_p be a solution of (1.1). Then the whole sequence $\{u_p\}$ converges to u_∞ strongly in $W^{1,m}(\Omega)$ as $p \to \infty$ for all $1 \le m < \infty$.*

In order to clarify the proof of Theorem 5.9 in the general case, we will make use of the following technical lemma.

Lemma 5.10. *Suppose that $F^*(x_0, .)$ satisfies the condition (A) about the strict convexity on its unit sphere. If*

$$\begin{cases} F^*(x_0, b) = F^*(x_0, a) > 0 \\ \langle \alpha \ \partial_\xi F^*(x_0, b) - \partial_\xi F^*(x_0, a), b - a\rangle = 0 \textit{ for some } \alpha \ge 0 \end{cases}$$

then

$$\partial_\xi F^*(x_0, b) \cap \partial_\xi F^*(x_0, a) \neq \emptyset$$

and

$$b = a.$$

Proof. First, let us show that $\partial_\xi F^*(x_0, b) \cap \partial_\xi F^*(x_0, a) \neq \emptyset$. Suppose, contrary to our claim, that $\partial_\xi F^*(x_0, b) \cap \partial_\xi F^*(x_0, a) = \emptyset$. Let $z_2 \in \partial_\xi F^*(x_0, b)$. We get that $z_2 \notin \partial_\xi F^*(x_0, a)$ and hence (thanks to Euler's Homogeneous Function Theorem, see [24, p. 173–174])

$$\langle z_2, a\rangle < F^*(x_0, a) = F^*(x_0, b) = \langle z_2, b\rangle. \tag{5.9}$$

In the same way, for any $z_1 \in \partial_\xi F^*(x_0, a)$, we get

$$\langle z_1, a\rangle > \langle z_1, b\rangle. \tag{5.10}$$

From (5.9) and (5.10), we get

$$\begin{cases} \langle z_2, b - a\rangle > 0 \\ \langle z_1, b - a\rangle < 0. \end{cases}$$

It follows that, for any $\alpha \ge 0$,

$$\alpha\langle z_2, b - a\rangle > \langle z_1, b - a\rangle$$

or equivalently,

$$\langle \alpha z_2 - z_1, b - a\rangle > 0,$$

which is a contradiction. This proves the first assertion of the lemma.

Now, let $z \in \partial_\xi F^*(x_0, b) \cap \partial_\xi F^*(x_0, a)$. We get $z \in \partial_\xi F^*(x_0, 0)$ and

$$\langle z, b\rangle = F^*(x_0, b) \text{ and } \langle z, a\rangle = F^*(x_0, a).$$

Since $F^*(x_0, b) = F^*(x_0, a) := l > 0$, we obtain that $\langle z, \frac{b}{l} - \frac{a}{l} \rangle = 0$. For any $t \in (0, 1)$, consider $v := (1-t)\frac{a}{l} + t\frac{b}{l}$. It is clear that

$$\langle z, v - \frac{a}{l} \rangle = \frac{t}{l}\langle z, b - a \rangle = 0.$$

It follows that, by $z \in \partial_\xi F^*(x_0, 0)$ if necessary,

$$1 \geq F^*(x_0, v) \geq \langle z, v \rangle = \langle z, \frac{a}{l} \rangle = \frac{F^*(x_0, a)}{l} = 1,$$

which implies $F^*(x_0, v) = 1$. Since $F^*(x_0, .)$ satisfies the assumption (A), we deduce that $b = a$. □

Proof of Theorem 5.9. We divide the proof into two main steps. Firstly, let us show that the whole sequence ∇u_p converges to ∇u_∞ almost-everywhere in Ω as $p \to \infty$. Indeed, take $u_p - u_\infty \in W_0^{1,p}(\Omega)$ as test functions in (1.1), we get

$$\int_\Omega \langle F^*(x, \nabla u_p)^{p-1} \partial_\xi F^*(x, \nabla u_p), \nabla(u_p - u_\infty) \rangle = \int_\Omega f(u_p - u_\infty).$$

Therefore, by the fact that $u_p \rightrightarrows u_\infty$ uniformly, $\nabla(u_p - u_\infty) \rightharpoonup 0$ weakly and $F^*(x, \nabla u_\infty(x)) = 1$ a.e. in Ω, we get that

$$\begin{aligned}&\int_\Omega \langle F^*(x, \nabla u_p)^{p-1} \partial_\xi F^*(x, \nabla u_p) - F^*(x, \nabla u_\infty)^{p-1} \partial_\xi F^*(x, \nabla u_\infty), \nabla(u_p - u_\infty) \rangle \\ &= \int_\Omega f(u_p - u_\infty) - \int_\Omega \langle \partial_\xi F^*(x, \nabla u_\infty), \nabla(u_p - u_\infty) \rangle \to 0 \quad \text{as} \quad p \to \infty.\end{aligned}$$

Let us consider the integrand

$$g_p := \langle F^*(x, \nabla u_p)^{p-1} \partial_\xi F^*(x, \nabla u_p) - F^*(x, \nabla u_\infty)^{p-1} \partial_\xi F^*(x, \nabla u_\infty), \nabla(u_p - u_\infty) \rangle.$$

Observe that $g_p \geq 0$ (by the convexity of $F^*(x, \xi)^p$ with respect to the second variable ξ) and thus g_p converges to 0 strongly in $L^1(\Omega)$. Then, up to a subsequence, g_p converges to 0 almost-everywhere in Ω. Let us set

$$\begin{aligned}Z := \{x \in \Omega : F^*(x, \nabla u_\infty(x)) = 1\} \bigcap \left\{x \in \Omega : \lim_{p\to\infty} g_p = 0\right\} \\ \bigcap \{x \in \Omega : F^*(x, .) \text{ satisfies the assumption (A)}\}.\end{aligned}$$

It is clear that $|Z| = |\Omega|$ (the Lebesgue measure of sets). We are now in a position to show that $\nabla u_p(x_0) \to \nabla u_\infty(x_0)$ for any $x_0 \in Z$. Let us fix any $x_0 \in Z$. Denote $\nabla u_p(x_0) = \xi_p$. By Euler's Homogeneous Function Theorem (see [24, p. 173–174]), we get $\langle \partial_\xi F^*(x_0, \xi_p), \xi_p \rangle = F^*(x_0, \xi_p)$ and

$$\begin{aligned}&g_p(x_0) \\ &= F^*(x_0, \xi_p)^p - \langle F^*(x_0, \xi_p)^{p-1} \partial_\xi F^*(x_0, \xi_p), \nabla u_\infty(x_0) \rangle - \langle \partial_\xi F^*(x_0, \nabla u_\infty), \xi_p - \nabla u_\infty \rangle.\end{aligned}$$

Since $g_p(x_0) \to 0$, $F^*(x_0, \xi_p)$ is bounded and so is ξ_p. Up to a subsequence, $\xi_p \to \xi$. Let us show that $F^*(x_0, \xi) = 1$. If $F^*(x_0, \xi) > 1$, then $F^*(x_0, \xi_p) > 1$ for large p and therefore $g_p(x_0) \to \infty$, a contradiction. If $F^*(x_0, \xi) < 1$, then as $p \to \infty$

$$\begin{aligned}g_p(x_0) \to &- \langle \partial_\xi F^*(x_0, \nabla u_\infty(x_0)), \xi - \nabla u_\infty(x_0) \rangle \\ &= -\langle \partial_\xi F^*(x_0, \nabla u_\infty(x_0)), \xi \rangle + F^*(x_0, \nabla u_\infty(x_0)) \\ &\geq F^*(x_0, \nabla u_\infty(x_0)) - F^*(x_0, \xi) > 0,\end{aligned}$$

which is again a contradiction.

So we have shown that $\nabla u_p(x_0) \to \xi$ and $F^*(x_0, \xi) = 1$. Since $g_p(x_0) \to 0$, we obtain

$$\langle \alpha \partial_\xi F^*(x_0, \xi) - \partial_\xi F^*(x_0, \nabla u_\infty(x_0)), \xi - \nabla u_\infty(x_0) \rangle = 0$$

for some $\alpha \geq 0$. Following Lemma 5.10, we conclude that $\xi = \nabla u_\infty(x_0)$. By the uniqueness of the limit (see Proposition 4.3), we deduce that the whole sequence ∇u_p converges to ∇u_∞ almost-everywhere in Ω as $p \to \infty$.

Secondly, the task is now to show the strong convergence of ∇u_p to ∇u_∞ in $L^m(\Omega)$ as $p \to \infty$ for any $1 \leq m < \infty$. Due to Hölder's inequality, it is enough to check for the case $m \geq 2$. As a consequence of Clarkson's inequality, we get

$$\int_\Omega |\nabla u_p - \nabla u_\infty|^m \leq 2^{m-1} \left(\int_\Omega |\nabla u_p|^m + \int_\Omega |\nabla u_\infty|^m \right).$$

By using (3.5), the non-degeneracy of F^* and Hölder's inequality, we get that $\{\nabla u_p\}$ is bounded in $L^m(\Omega)$ independent of p for any $p \geq m$, which implies that $\int_\Omega |\nabla u_p - \nabla u_\infty|^m$ is bounded by a constant independent of p. It follows from the reverse Fatou lemma that

$$\limsup_{p\to\infty} \int_\Omega |\nabla u_p - \nabla u_\infty|^m \leq \int_\Omega \limsup_{p\to\infty} |\nabla u_p - \nabla u_\infty|^m = 0,$$

where the last equality is due to the above almost-everywhere convergence.

Consequently, ∇u_p converges to ∇u_∞ strongly in $L^m(\Omega)$ as $p \to \infty$. It follows that the whole sequence u_p converges to u_∞ strongly in $W^{1,m}(\Omega)$ as $p \to \infty$ for any $1 \leq m < \infty$, and the proof is complete. ☐

Acknowledgments

This research is funded by Vietnam Ministry of Education and Training (through National Key Program for the Development of Mathematics in the 2021–2030 period) under grant number B2023-CTT-02.

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