Tutorial

September 17, 2026

1 Part I: Matrix Analysis

Exercise 1

a. Justify why a rank-one matrix A can always be written $A = uv^\top$.

b. Express the matrix $B = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 10 \\ 0 & 0 & 0 \end{bmatrix}$ as the product of two vectors: $B = uv^\top$.

c. Compute the eigenvalues and associated eigenvectors of B.

d. Justify why B has rank 1.

Exercise 2 ⋆

Let A be the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 2 & 2 & 3 & 3 \end{bmatrix}.$$

a. Determine whether the system $Ax = 0$ has zero, one or infinitely many solutions, and compute a basis of the space of solutions.

b. Is it true that the system $Ax = b$ has a solution for any $b \in \mathbb{R}^3$? If so, prove the statement; otherwise, find a counterexample.

Exercise 3 ⋆

Let M be a 2×2 matrix with real coefficients and eigenvalues 3 and 5, with eigenvectors $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -6 \end{bmatrix}$ respectively.

a. Compute $M \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

b. Find a diagonal matrix D and two matrices A, A^{-1} (each the inverse of the other) such that $M = ADA^{-1}$.

Exercise 4 ⋆

Compute the distance between

$$x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad y = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$$

induced by the inner products i) $\langle x, y \rangle = x^\top y$, ii) $\langle x, y \rangle_Q = x^\top Q y$ with $Q = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix}$.

Exercise 5 ⋆

Compute the angle between

$$x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad y = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

using i) $\langle x, y\rangle = x^\top y$, ii) $\langle x, y\rangle_Q = x^\top Q y$ with $Q = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$.

Exercise 6 ⋆

Let $M \in \mathcal{M}_3(\mathbb{R})$ be the matrix

$$M = \begin{bmatrix} -2 & 0 & 1 \\ -2 & 0 & 1 \\ -4 & 0 & 2 \end{bmatrix}.$$

Compute the eigenvalues and the eigenvectors of M. Is M diagonalizable?

Exercise 7

Recall that the Frobenius norm of a matrix $A \in \mathbb{R}^{n\times n}$ is defined as $\|A\|_{\mathsf{F}} = \sqrt{\mathrm{Tr}(A^\top A)}$, where Tr denotes the trace of a matrix, i.e., the sum of its diagonal entries.

a. Show that

$$\|A\|_{\mathsf{F}} = \Big(\sum_{i,j} |A_{ij}|^2\Big)^{1/2}.$$

Thus the Frobenius norm is simply the Euclidean norm of the matrix when it is considered as an element of $\mathbb{R}^{n^2}$. Note also that the Frobenius norm is much easier to compute than the spectral norm (i.e., the largest singular value).

b. Show that if U and V are orthogonal, then $\|UA\|_{\mathsf{F}} = \|AV\|_{\mathsf{F}} = \|A\|_{\mathsf{F}}$. Thus the Frobenius norm is invariant under left or right multiplication by an orthogonal matrix.

c. Show that $\|A\|_{\mathsf{F}} = \sqrt{\sigma_1^2 + \cdots + \sigma_r^2}$, where $\sigma_1, \ldots, \sigma_r$ are the nonzero singular values of A. Then show that $\sigma_{\mathsf{max}}(A) \leq \|A\|_{\mathsf{F}} \leq \sqrt{r}\,\sigma_{\mathsf{max}}(A)$. In particular, $\|Ax\| \leq \|A\|_{\mathsf{F}}\|x\|$ for all x.

Exercise 8 ⋆

Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

⋆ a. Compute the eigenvalues of A. Is the matrix invertible?

(The following questions can be treated for an arbitrary matrix A.)

⋆ b. Let $B = AA^\top$. Check that B is symmetric. What can you say about its eigenvalues? Compute the eigenvalues of B.

⋆ c. Without further computation, give the eigenvalues of $C = A^\top A$.

d. Give conditions for C to be invertible. In particular, show that if A has full column rank, then C is invertible.

e. Show that C^{-1} is symmetric whenever it exists. What can you then say about its eigenvalues?

2 Part II: Optimization

Exercise 9 ⋆ Basic differential calculus

Compute the gradient of f (or its Jacobian matrix J_f if f is vector-valued):

- $f : x \in \mathbb{R}^n \mapsto u^\top x$, with $u \in \mathbb{R}^n$.
- $f : x \in \mathbb{R}^n \mapsto \frac{1}{2}x^\top Ax + b^\top x + c$, with $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Same question with $A \in \mathcal{S}^n$ (symmetric matrices).
- $f : x \in \mathbb{R}^n \mapsto Ax + b \in \mathbb{R}^m$, with $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$.
- $f : x \in \mathbb{R}^n \mapsto \|Ax - b\|_2^2$.
- $f(x) = \|x\|_2$, for $x \neq 0$.
- $f(x) = \log\left(\sum_{i=1}^m \exp(a_i^\top x + b_i)\right)$, with $a_1, \dots, a_m \in \mathbb{R}^n$, $b_1, \dots, b_m \in \mathbb{R}$.

Exercise 10 ⋆ Fundamentals of convexity

This exercise proves and illustrates some results seen in the course.

⋆ a. Let f and g be two convex functions. Show that $m(x) = \max(f(x), g(x))$ is convex.

⋆ b. Show that $f_1(x) = \max(x^2 - 1, 0)$ is convex.

⋆ c. Let f be a convex function and g be a convex, non-decreasing function. Show that $c(x) = g(f(x))$ is convex.

⋆ d. Show that $f_2(x) = \exp(x^2)$ is convex. What about $f_3(x) = \exp(-x^2)$?

e. Consider the function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ defined by

$$f(x) = \begin{cases} -\ln(1 - \|x\|) & \text{if } \|x\| < 1, \\ +\infty & \text{otherwise.} \end{cases}$$

Show that f is convex.

f. Justify why the 1-norm, the 2-norm and the squared 2-norm are convex.

Exercise 11 Strict and strong convexity

A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be

- *strictly convex* if for any $x \neq y \in \mathbb{R}^n$ and any $\alpha \in (0, 1)$,
$$f(\alpha x + (1-\alpha)y) < \alpha f(x) + (1-\alpha)f(y);$$
- *strongly convex* if there exists $\beta > 0$ such that $f - \frac{\beta}{2}\|\cdot\|_2^2$ is convex.

a. For a strictly convex function f, show that the problem

$$\begin{cases} \min f(x) \\ x \in C \end{cases}$$

where C is a convex set, admits at most one solution.

b. Show that a strongly convex function is also strictly convex.
Hint: use the identity $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x - y\|^2$.

c. Let f be a twice differentiable function. Show that f is strongly convex if and only if there exists $\beta > 0$ such that the eigenvalues of $\nabla^2 f(x)$ are greater than or equal to β for all x.

Exercise 12 ⋆ Optimality conditions

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a twice differentiable function and $\bar{x} \in \mathbb{R}^n$. We suppose that f admits a local minimum at $\bar{x}$, that is, $f(x) \geq f(\bar{x})$ for all x in a neighborhood[1] of $\bar{x}$.

a. For any direction $u \in \mathbb{R}^n$, we define the function $q : \mathbb{R} \to \mathbb{R}$ by $q(t) = f(\bar{x} + tu)$. Compute $q'(t)$.

b. Using the first-order Taylor expansion of q at 0, show that $\nabla f(\bar{x}) = 0$.

c. Compute $q''(t)$. Using the second-order Taylor expansion of q at 0, show that $\nabla^2 f(\bar{x})$ is positive semi-definite.

Exercise 13 ⋆ Descent lemma

A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be L-smooth if it is differentiable and its gradient ∇f is L-Lipschitz continuous, that is,

$$\forall x, y \in \mathbb{R}^n, \quad \|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|.$$

The goal of the exercise is to prove that if $f : \mathbb{R}^n \to \mathbb{R}$ is L-smooth, then for all $x, y \in \mathbb{R}^n$,

$$f(x) \leq f(y) + (x - y)^\top \nabla f(y) + \frac{L}{2}\|x - y\|^2.$$

a. Starting from the fundamental theorem of calculus, which states that for all $x, y \in \mathbb{R}^n$,

$$f(x) - f(y) = \int_0^1 (x - y)^\top \nabla f(y + t(x - y))\, \mathrm{d}t,$$

prove the descent lemma.

b. Give a function for which the inequality is tight and one for which it is not.

Exercise 14 Smooth functions

Consider the constant stepsize gradient algorithm $x_{k+1} = x_k - \gamma \nabla f(x_k)$ applied to an L-smooth function f admitting a minimizer (i.e., some $x^\star$ such that $f(x) \geq f(x^\star)$ for all x).

⋆ a. Use the *descent lemma* to prove the convergence of the sequence $(f(x_k))$ when $\gamma \leq 2/L$.

b. Does the sequence (x_k) converge? To what?

[1] Formally: for all $x \in \mathbb{R}^n$ such that $\|x - \bar{x}\| \leq \varepsilon$, for some $\varepsilon > 0$ and some norm $\|\cdot\|$.