Tutorial : Optimization

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Exercice 1 Basic Differential calculus

Compute the gradients of :

a. $f_1(x) = u^T x$.

b. $f_2(x) = x^T A x$.

c. $f_3(x) = \|Ax - b\|_2^2$.

d. $f_4(x) = \|x\|_2$.

Exercice 2 Fundamentals of convexity

This exercise proves and illustrates some results seen in the course.

a. Let f and g be two convex functions. Show that $m(x) = \max(f(x), g(x))$ is convex.

b. Show that $f_1(x) = \max(x^2 - 1, 0)$ is convex.

c. Let f be a convex function and g be a convex, non-decreasing function. Show that $c(x) = g(f(x))$ is convex.

d. Show that $f_2(x) = \exp(x^2)$ is convex. What about $f_3(x) = \exp(-x^2)$?

e. Consider the function $f = \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ defined by

$$f(x) = \begin{cases} -\ln(1 - \|x\|) \text{ if } \|x\| < 1 \\ +\infty \text{ otherise.} \end{cases}$$

Show that f is convex.

f. Justify why the 1-norm, the 2 norm, and the squared 2-norm are convex.

Exercice 3 Strict and strong convexity

A function $f : \mathbb{R}^n \to \mathbb{R}$ is said

- *strictly convex* if for any $x \neq y \in \mathbb{R}^n$ and any $\alpha \in]0, 1[$

$$f(\alpha x + (1 - \alpha)y) < \alpha f(x) + (1 - \alpha)f(y)$$

- *strongly convex* if there exists $\beta > 0$ such that $f - \frac{\beta}{2}\| \cdot \|_2^2$ is convex.

a. For a strictly convex function f, show that the problem

$$\begin{cases} \min f(x) \\ x \in C \end{cases}$$

where C is a convex set admits at most one solution.

b. Show that a strongly convex function is also strictly convex.
Hint : use the identity $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2$.

c. Let f be a twice differentiable function. Show that f is strongly convex if and only if there exists $\beta > 0$ such that the eigenvalues of $\nabla^2 f(x)$ are larger than β for all x.

Exercice 4 Optimality conditions

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a twice differentiable function and $\bar{x} \in \mathbb{R}^n$. We suppose that f admits a local minimum at $\bar{x}$ that is $f(x) \geq f(\bar{x})$ for all x in a neighborhood[1] of $\bar{x}$.

a. For any direction $u \in \mathbb{R}^n$, we define the $\mathbb{R} \to \mathbb{R}$ function $q(t) = f(\bar{x} + tu)$. Compute $q'(t)$.

b. By using the first order Taylor expansion of q at 0, show that $\nabla f(\bar{x}) = 0$.

c. Compute $q''(t)$. By using the second order Taylor expansion of q at 0, show that $\nabla^2 f(\bar{x})$ is positive semi-definite.

Exercice 5 Descent lemma

A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be L-smooth if it is differentiable and its gradient ∇f is L-Lipchitz continuous, that is

$$\forall x, y \in \mathbb{R}^n, \quad \|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|.$$

The goal of the exercise is to prove that if $f : \mathbb{R}^n \to \mathbb{R}$ is L-smooth, then for all $x, y \in \mathbb{R}^n$,

$$f(x) \leq f(y) + (x-y)^\mathsf{T}\nabla f(y) + \frac{L}{2}\|x-y\|^2$$

a. Starting from fundamental theorem of calculus stating that for all $x, y \in \mathbb{R}^n$,

$$f(x) - f(y) = \int_0^1 (x-y)^\mathsf{T}\nabla f(y + t(x-y))\mathrm{d}t$$

prove the descent lemma.

b. Give a function for which the inequality is tight and one for which it is not.

Exercice 6 Smooth functions

Consider the constant stepsize gradient algorithm $x_{k+1} = x_k - \gamma\nabla f(x_k)$ on an L-smooth function f with some minimizer (i.e. some $x^\star$ such that $f(x) \geq f(x^\star)$ for all x).

a. Use the *descent lemma* to prove convergence of the sequence $(f(x_k))$ when $\gamma \leq 2/L$.

b. Does the sequence (x_k) converge ? To what ?

[1] Formally, one would write $\forall x \in \mathbb{R}^n$ such that $\|x - \bar{x}\| \leq \varepsilon$ for $\varepsilon > 0$ and some norm $\|\cdot\|$.