Tutorial : Matrix Analysis

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Exercice 1

a. Justify why a rank 1 matrix A can always be written $A = uv^\top$.

b. Express matrix $B = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 10 \\ 0 & 0 & 0 \end{bmatrix}$ as the product of two vectors : $B = uv^\top$.

c. Compute eigenvalues and associated eigenvectors of matrix B.

d. Justify why B is rank 1.

Exercice 2

Let A be the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 2 & 2 & 3 & 3 \end{bmatrix}$$

a. Determine if the system $Ax = 0$ has zero, one or infinitely many solutions, and compute a basis of the space of solutions.

b. Is it true that the system $Ax = b$ has a solution for any $b \in \mathbb{R}^3$? If so, prove the statement, otherwise find a counterexample.

Exercice 3

Let M be a 2×2 matrix with real coefficients and eigenvalues 3 and 5 , with eigenvectors $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -6 \end{bmatrix}$ respectively.

a. Compute $M \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

b. Find a diagonal matrix D and two matrices A, A^{-1} (each inverse of the other) such that $M = ADA^{-1}$.

Exercice 4

Let $M \in \mathcal{M}_3(\mathbb{R})$ be the matrix

$$M = \begin{bmatrix} -2 & 0 & 1 \\ -2 & 0 & 1 \\ -4 & 0 & 2 \end{bmatrix}.$$

Compute the eigenvalues and the eigenvectors of M. Is M diagonalizable ?

Exercice 5

Recall that the Frobenius norm of a matrix $A \in \mathbf{R}^{n\times n}$ is defined as $\|A\|_{\mathsf{F}} = \sqrt{\operatorname{Tr} A^T A}$. (Recall Tr is the trace of a matrix, i.e., the sum of the diagonal entries.)

a. Show that

$$\|A\|_{\mathsf{F}} = \left(\sum_{i,j} |A_{ij}|^2\right)^{1/2}.$$

Thus the Frobenius norm is simply the Euclidean norm of the matrix when it is considered as an element of $\mathbf{R}^{n^2}$. Note also that it is much easier to compute the Frobenius norm of a matrix than the (spectral) norm (i.e., maximum singular value).

b. Show that if U and V are orthogonal, then $\|UA\|_{\mathsf{F}} = \|AV\|_{\mathsf{F}} = \|A\|_{\mathsf{F}}$. Thus the Frobenius norm is not changed by a pre- or post- orthogonal transformation.

c. Show that $\|A\|_{\mathsf{F}} = \sqrt{\sigma_1^2 + \cdots + \sigma_r^2}$, where $\sigma_1, \ldots, \sigma_r$ are the singular values of A. Then show that $\sigma_{\mathsf{max}}(A) \leq \|A\|_{\mathsf{F}} \leq \sqrt{r}\sigma_{\mathsf{max}}(A)$. In particular, $\|Ax\| \leq \|A\|_{\mathsf{F}}\|x\|$ for all x.

Exercice 6

Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

a. Compute the eigenvalues of A. Is the matrix invertible ?

(the following questions can be treated for an arbitrary matrix A)

b. Let $B = AA^\top$. Check that B is symmetric. What can you say about its eigenvalues ? Compute the eigenvalues of B.

c. Without further computation, give the eigenvalues of $C = A^\mathsf{T} A$.

d. Give conditions for C to be invertible. Notably, show that if A is full column rank, then C is also full column rank.

e. Show that C^{-1} is symmetric whenever it exists. What can you then say about its eigenvalues ?