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QUASI-CONVEX HAMILTON–JACOBI EQUATIONS VIA FINSLER p-LAPLACE–TYPE OPERATORS*

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Abstract. In this paper we show that the maximal viscosity solution of a class of quasi-convex Hamilton–Jacobi equations, coupled with inequality constraints on the boundary, can be recovered by taking the limit as $p \to \infty$ in a family of Finsler p-Laplace problems. The approach also enables us to provide an optimal solution to a Beckmann-type problem in the general Finslerian setting and allows recovering a bench of known results based on the Evans–Gangbo technique.

Key words. Hamilton–Jacobi equation, p-Laplace operator, Finsler structure, Beckman problem

MSC codes. 35F21, 35A15

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1. Introduction.

Let Ω be a smooth bounded subset of $\mathbb{R}^N$. Consider a continuous Hamiltonian $F : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}$ such that, for all $x \in \overline{\Omega}$,

- $Z(x) := \{\xi \in \mathbb{R}^N : \ F(x, \xi) \leq 0\}$ is a convex and compact subset of $\mathbb{R}^N$,
- $0 \in \text{int}(Z(x))$.

Our main aim concerns the Hamilton–Jacobi (HJ for short) equation of first order:

$$F(x, \nabla u) = 0 \text{ in } \Omega. \tag{1.1}$$

The class of HJ PDE is central in several branches of mathematics, both from theoretical, numerical, and application points of view. The applications in classical mechanics, optics, Hamiltonian dynamics, semiclassical quantum theory, Riemannian and Finsler geometry, and the optimal control theory are very important.

In addition to its connection with Hamilton's equations, in the case where the Hamiltonian has sufficient regularity, further connection with common PDEs was established in the literature. For instance, it appears in the classical limit of the Schrödinger equation (see, e.g., [1]). Its connection with the discount HJ equation $\lambda u + F(x, \nabla u) = 0$ as $\lambda \to 0$ was established in the seminal paper [22] and generalized in [9]. The vanishing viscosity method for first order HJ equations establishes the connection of HJ equations with the second order PDE $-\epsilon\Delta u + F(x, \nabla u) = 0$ as $\epsilon \to 0$ (see, for instance, [7, 21]). The celebrated paper of Varadhan [29] shows that the heat kernel in a Riemannian manifold can be approximated by a Gaussian kernel and thus makes the link between the heat equation and the HJ equation. This connection can be also done via Hopf–Cole transformation as showed in [6]. This kind of transformation also allows recovering the HJ equation in the large-scale hyperbolic limit of a class of kinetic equation (see, e.g., [5]).

Recently, the connection between the HJ equation, optimal mass transport and Beckmann's problem was established in [12, 13] with a flavor of variational approach.

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In particular, these connections work out a nonlinear divergence-form PDE, called the Monge–Kantorovich equation, that we can associate definitively with the HJ equation. The connection is not straightforward since the optimal mass transportation, Beckmann's problem, and the associate divergence formulation are not standard. Roughly speaking, the offset is connected to some unknown distribution of mass concentrated on the boundary which would both counterbalance the involved optimal mass transportation phenomena and describe the normal trace of the allowed flux in the divergence formulation (see [12, 13] for the details). The approach blends sophisticated tools from variational analysis, convex duality, and trace-like operator for the so-called divergence-measure field. To strengthen the connection with the divergence equation and to shape the "pretending diffusive taste" of HJ equation, we propose in this paper how to achieve the solutions of the HJ equation using an elliptic PDE of Finsler p-Laplace type. The Finsler structure associated with the Hamiltonian F takes part in the PDE in a common way bringing out some kind of anisotropic p-Laplace PDE that we call here the Finsler p-Laplace equation. We treat the equation (1.1) with a double obstacle on the boundary. Moreover, thanks to the substantial link of the HJ equation with the optimal mass transport, as well as the Beckmann problem, these problems will be concerned in their turn with the approach using the Finsler p-Laplace equation.

To describe roughly the approach, we consider the peculiar case of eikonal equation with Dirichlet boundary condition:

$$\begin{cases} |\nabla u| = k & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where k is a positive continuous function in $\overline{\Omega}$ and $\partial\Omega$ denotes the boundary of Ω. It is well known by now that the intrinsic distance defined by

$$d_k(x,y) := \inf_{\zeta\in\Gamma(x,y)} \int_0^1 k(\zeta(t))\,|\dot{\zeta}(t)|\mathrm{d}t,$$

where $\Gamma(x,y)$ is the set of Lipchitz curves joining x and y, describes the maximal viscosity subsolution through the following formula:

$$u(x) = \min_{y\in\partial\Omega} \{d_k(y,x) + g(y)\}\,. \tag{1.3}$$

Here $g : \partial\Omega \to \mathbb{R}$ is assumed to be a continuous function satisfying the compatibility condition

$$g(x) - g(y) \le d_k(y,x) \ \text{ for all } x, y \in \partial\Omega.$$

Since (1.3) is likewise the unique solution of the maximization problem

$$\max_{z\in W^{1,\infty}(\Omega)} \left\{ \int_\Omega z(x)\mathrm{d}x :\ |\nabla z(x)| \le k(x) \text{ and } z = g \text{ on } \partial\Omega \right\}, \tag{1.4}$$

we know (see [12, 13]) that a dual problem for (1.4) reads

(1.5)

$$\min_{\phi\in\mathcal{M}_b(\overline{\Omega})^N,\, \nu\in\mathcal{M}_b(\partial\Omega)} \left\{ \int_{\overline{\Omega}} k\,\mathrm{d}|\phi| + \int_{\partial\Omega} g\mathrm{d}\nu :\quad -\operatorname{div}(\phi) = \chi_\Omega - \nu \text{ in } \ \mathcal{D}^{'}(\mathbb{R}^N) \right\},$$

which constitutes actually a new variant of Beckmann's problem with boundary cost g. Here $\mathcal{M}_b$ is used to denote the set of finite Radon measures. In particular, this is connected to the Monge optimal mass transport problem

$$\inf\left\{\int_\Omega d_k(x,T(x))\mathrm{d}x \;:\; \nu\in\mathcal{M}_b(\partial\Omega),\ T_\sharp\chi_\Omega=\nu\right\},$$

as well as to the Monge–Kantorovich relaxed problem

$$\min\left\{\int_{\Omega\times\Omega} d_k(x,y)\mathrm{d}\gamma(x,y)\;:\nu\in\mathcal{M}_b(\partial\Omega),\ \gamma\in\mathcal{M}^+(\Omega\times\Omega),\ (\pi_x)_\sharp\gamma=\chi_\Omega,(\pi_y)_\sharp\gamma=\nu\right\}.$$

Even if here the so-called target measure ν is an unknown parameter of the problem, one sees that the problem aims certainly an optimal mass transportation between $\rho_1:=\chi_\Omega$ and $\rho_2:=\nu$, and moreover u, given by (1.3) (the unique solution of (1.2)), is an Kantorovich potential of transportation. Since the pioneering work of Evans and Gangbo (cf. [14]) in the case where $k\equiv 1$, it is known that key information concerning u may be given by the uniform limit of u_p, the solution of the modified p-Laplace equation

$$\begin{cases} -\Delta_p\left(\frac{u_p}{k}\right)=\rho_1-\rho_2 & \text{in } \overline{\Omega},\\ u=g & \text{on } \partial\Omega.\end{cases}\tag{1.6}$$

Following the results of [14], one can guess this limit to be given by the so-called Monge–Kantorovich system:

$$\begin{cases} -\operatorname{div}(\Phi)=\rho_1-\rho_2,\ |\nabla u|\le k & \text{in } \overline{\Omega},\\ \Phi=m\,\nabla u,\ m\ge 0,\ m(|\nabla u|-k)=0 & \text{a.e.},\\ u=g & \text{on } \partial\Omega.\end{cases}\tag{1.7}$$

Notice here that, apart from a few special cases out of the scope of our situation (cf. [26, Chapter 4.3] for discussions and references about regularity properties of Φ under extra assumptions), in general the flux Φ is a vector-valued measure, and it is closely connected to the solution of Beckmann problem (1.5). Coming back to the HJ equation (1.2), it is clear now that the Monge–Kantorovich system is a suitable divergence equation for the solution of (1.2). Moreover, the limit of the flux of (1.6) converges weakly to Φ picturing thereby some kind of "nonlinear diffusion" phenomena behind the HJ equation.

Contributions. In this paper, we are interested in studying the connection between the HJ equation, coupled with inequality constraints on the boundary,

$$\begin{cases} F(x,\nabla u)=0 & \text{in } \Omega,\\ \phi\le u\le\psi & \text{on } \partial\Omega,\end{cases}\tag{1.8}$$

and an elliptic problem of Finsler p-Laplace type that we will introduce below.

We show how to recover the maximal viscosity subsolution to the class of HJ equations of the type (1.8) using a family of Finsler p-Laplace problems (with boundary obstacles) as $p\to\infty$. Moreover, since the solution of (1.8) is intimately linked to the so-called Kantorovich–Rubinstein problem in optimal transport, an appropriate Beckmann transportation problem is derived, and its solution is provided. Essentially, this will be the content of Theorem 3.2 whose proof relies on the results and estimates of Propositions 2.3 and 3.1. Finally, we show in Proposition 4.1 that the limit as $p \to \infty$ of solutions of the p-Laplace problems is a Kantorovich potential for a classical Kantorovich problem involving the normal trace on the boundary of the optimal flow of Beckmann's problem. Our work illustrates some kind of "nonlinear diffusion" phenomena behind the HJ equation.

Related works. Concerning limits as $p \to \infty$ for the p-Laplace equations, one of the first mathematical studies is [2] with particular interest in torsional problems and ∞-harmonic functions, followed by the celebrated work of Evans and Gangbo [14]. Similar problems were considered in [16, 17] for transport problems with masses supported on the boundary. Variants of Monge–Kantorovich problems with boundary costs were addressed in [23], where the boundary costs can be seen as some import/export taxes. In the same spirit, similar results were obtained in [10] with some weighted Euclidean distance as a cost. The use of PDE techniques à la Evans and Gangbo in the Finsler framework was addressed recently in [18]. It is well known that Finsler metrics generalize the Riemannian ones and are of main interest in the study of optimal transport and minimal flow problems since they allow considering anisotropy, obstacles, etc..

Our work adds to these series of papers linking HJ equations to other PDEs, thanks to the variational approach (cf. [12]), and permits generalizing the works on mass transport recalled above. It shows once again the flexibility of the Evans–Gangbo method.

The rest of this paper is organized as follows: in section 2, we present assumptions and preliminary results concerning the notion of solution to the HJ equation coupled with obstacles on the boundary under consideration and Finsler p-Laplace equations, as well as their existence and characterization of solutions. In section 3, we derive suitable estimates independent of p and show the convergence of Finsler p-Laplace equations as $p \to \infty$. The existence and characterization of solutions to the limited variational problems are also studied in detail. Finally, the connection between the limited variational problems and a variant of Monge–Kantorovich transportation problem is derived in section 4.

2. Preliminaries.

2.1. Maximal viscosity subsolution. Consider the HJ equation of first order, coupled with some inequality constraints on the boundary

$$\begin{cases} F(x, \nabla u) = 0 & \text{in } \Omega, \\ \phi \le u \le \psi & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Here, $\phi, \psi \in C(\partial\Omega)$ satisfy the compatibility condition

$$\phi(x) - \psi(y) \le d_\sigma(y, x) \text{ for all } x, y \in \partial\Omega,$$

with d_σ being the intrinsic metric associated to F (see below).

For each $x \in \overline{\Omega}$, we define the support function $\sigma(x, .)$ of the 0-sublevel set of F by

$$\sigma(x, q) = \sup_{p \in Z(x)} \langle p, q \rangle \quad \text{for all } q \in \mathbb{R}^N,$$

which turns to be a Finsler metric (see subsection 2.2 below). Then, the intrinsic distance associated to F is defined through

$$d_\sigma(x,y) := \inf_{\zeta\in\Gamma(x,y)} \int_0^1 \sigma(\zeta(t),\dot{\zeta}(t))\mathrm{d}t,$$

where $\Gamma(x,y)$ is the set of Lipchitz curves joining x and y. In the case where $\phi\equiv\psi = g:\partial\Omega\to\mathbb{R}$ is a continuous function satisfying the compatibility condition

$$g(x)-g(y)\le d_\sigma(y,x) \quad \text{for all } x,y\in\partial\Omega,$$

it is well known (see, e.g., [15, 21]) that the maximal viscosity subsolution of

$$\begin{cases} F(x,\nabla u)=0 & \text{in } \Omega,\\ u=g & \text{on } \partial\Omega \end{cases} \tag{2.2}$$

is given by

$$u(x) = \min_{y\in\partial\Omega}\left\{d_\sigma(y,x)+g(y)\right\}. \tag{2.3}$$

Moreover, this solution coincides with the maximal volume solution. Indeed, using the fact that the set of all viscosity subsolutions of (2.2) coincides with the set of Lipschitz functions u satisfying

$$\sigma^*(x,\nabla u(x))\le 1 \text{ a.e.},$$

where σ^* is the dual of the support function σ defined through

$$\sigma^*(x,q) = \sup_{\sigma(x,p)\le 1}\langle p,q\rangle,$$

we proved in [12] that (2.3) is the unique solution of the following maximization problem:

$$\max_{z\in W^{1,\infty}(\Omega)}\left\{\int_\Omega z(x)\mathrm{d}x,\ \sigma^*(x,\nabla z(x))\le 1 \text{ and } z=g \text{ on } \partial\Omega\right\}.$$

Now, for the study of the general problem (2.1) with inequality constraints on the boundary, we make use of a similar notion of solution. Actually we have the following proposition.

PROPOSITION 2.1. *Under the assumption* (2.13), (2.1) *has a unique solution u in the sense of maximal volume; that is, u is the unique solution to the following maximization problem:*

$$\max_{z\in W^{1,\infty}(\Omega)}\left\{\int_\Omega z(x)\mathrm{d}x,\ \sigma^*(x,\nabla z(x))\le 1 \textit{ and } \phi\le z\le\psi \textit{ on } \partial\Omega\right\}. \tag{2.4}$$

Moreover, u is the maximal viscosity subsolution satisfying $\phi\le u\le\psi$ on $\partial\Omega$.

Remark 2.2. Let us say a few words about Perron's method for (2.1). First, let us denote by w the so-called Perron's solution of (2.1) given by

$$w(x) = \sup_{v\in K_{d_\sigma}}\{v(x)\},$$

where

$$K_{d_\sigma} = \{v \in W^{1,\infty}(\Omega) : \ \mathrm{Lip}_{d_\sigma}(v) \le 1 \text{ and } \phi \le z \le \psi \text{ on } \partial\Omega\},$$
$$\mathrm{Lip}_{d_\sigma}(v) = \sup_{\substack{x,y\in\Omega \\ x\neq y}} \left\{ \frac{v(y)-v(x)}{d_\sigma(x,y)} \right\},$$

and d_σ is the intrinsic distance associated to F which is recalled below in (2.12). Since K_{d_σ} coincides (see, e.g., [12]) with the set

$$K_{\sigma^*} = \{z \in W^{1,\infty}(\Omega) : \ \sigma^*(x, \nabla z(x)) \le 1 \text{ and } \phi \le z \le \psi \text{ on } \partial\Omega\},$$

one can easily show that the Perron solution w is the maximal volume solution u of the problem (2.4). Indeed, we have that $u \le w$. In addition, if we suppose that $u(x_0) < w(x_0)$ for some $x_0 \in \overline{\Omega}$, then we still have $u(x) < w(x)$ for any $x \in B(x_0, \epsilon)$ for some $\epsilon > 0$. Then taking $z = \max(u, w)$ we have that $\int_\Omega z\mathrm{d}x > \int_\Omega u\mathrm{d}x$ which contradicts the fact that u has maximal volume.

2.2. Finsler p-Laplacian equation. Let Ω be a bounded open subset of $\mathbb{R}^N$; a Finsler metric is a continuous function $H : \overline{\Omega}\times\mathbb{R}^N \to [0,\infty)$ such that $H(x,.)$ is convex and positively 1−homogeneous in the second variable, that is, $H(x, tp) = tH(x, p)$ for every $t \ge 0$.

We define the dual of a Finsler metric H (which is also a Finsler metric) by

$$H^*(x,q) = \sup_{H(x,p)\le 1} \langle p, q\rangle = \sup_{p\neq 0} \frac{\langle p,q\rangle}{H(x,p)}.$$

In this paper, we assume that H is a nondegenerate Finsler metric; that is, there exist $a, b > 0$ such that

$$a|p| \le H(x,p) \le b|p| \tag{2.5}$$

for all $(x,p) \in \overline{\Omega} \times \mathbb{R}^N$. In other words, one has

$$\tilde{a}|q| \le H^*(x,q) \le \tilde{b}|q| \tag{2.6}$$

for some $\tilde{a}, \tilde{b} > 0$. Moreover, we have the Cauchy-Schwarz–like inequality

$$\langle p,q\rangle \le H(x,p)H^*(x,q). \tag{2.7}$$

Euler's homogeneous function theorem (see, e.g., [24]) says that

$$\partial_\xi H^*(x,p)\cdot p = H^*(x,p) \ \text{ for any } \ p \in \mathbb{R}^N, \tag{2.8}$$

and by convexity of H^*, we have

$$\partial_\xi H^*(x,p)\cdot q \le H^*(x,q) \ \text{ for any } \ p,q \in \mathbb{R}^N.$$

Thus, using (2.6) we get

$$|\partial_\xi H^*(x,p)\cdot q| \le \tilde{b}|q| \ \text{ for any } \ p,q \in \mathbb{R}^N. \tag{2.9}$$

Finally, we have

$$H(x, \partial_\xi H^*(x,p)) = 1 \quad \text{for any } p \in \mathbb{R}^N. \tag{2.10}$$

For details and additional properties we refer the reader to [27].

Every Finsler metric induces a Finsler distance via the so-called length (or action) functional. The action of a Lipschitz curve $\xi \in \mathrm{Lip}([0,1];\overline{\Omega})$ is defined through

$$A_H(\xi) = \int_0^1 H(\xi(s), \dot{\xi}(s))\mathrm{d}s. \tag{2.11}$$

The induced distance d_H by the action functional (2.11) reads as

$$d_H(x,y) = \inf_{\xi \in \Gamma(x,y)} A_H(\xi). \tag{2.12}$$

Note that, in general, $H(x,p)$ is not even in p so that d_H may be nonsymmetric; i.e., it may happen that $d_H(x,y) \neq d_H(y,x)$.

Assuming that $H^*(x,.) \in C^1(\mathbb{R}^N \setminus \{0\})$ and the compatibility condition

$$\phi(x) - \psi(y) \leq d_H(y,x) \text{ for all } x,y \in \partial\Omega, \tag{2.13}$$

we consider the following Finsler (also called anisotropic) p-Laplace problems:

$$\begin{cases} -\operatorname{div}(H^*(x,\nabla u_p)^{p-1}\partial_\xi H^*(x,\nabla u_p)) = \rho & \text{in } \Omega, \\ \phi \leq u_p \leq \psi & \text{on } \partial\Omega, \end{cases} \tag{2.14}$$

where $p > N$ and $\rho \in L^2(\Omega)$ are given and $\partial_\xi H^*$ stands for the derivative of H^* with respect to the second variable. To study this problem let us consider the set

$$\mathcal{W}_{\phi,\psi} = \{u \in W^{1,p}(\Omega) : \ \phi \leq u \leq \psi \text{ on } \partial\Omega\},$$

and we denote by

$$\Theta_p = H^*(x,\nabla u_p)^{p-1}\partial_\xi H^*(x,\nabla u_p).$$

Proposition 2.3. *Assume* (2.13) *is strict, that is,*

$$\phi(x) - \psi(y) < d_H(y,x) \textit{ for all } x,y \in \partial\Omega. \tag{2.15}$$

The problem (2.14) *has a unique solution u_p in the following sense: $u_p \in \mathcal{W}_{\phi,\psi}$ and*

$$\int_\Omega \Theta_p \cdot \nabla(u_p - \xi)\,\mathrm{d}x \leq \int_\Omega \rho\,(u_p - \xi)\,\mathrm{d}x \quad \textit{for any } \xi \in \mathcal{W}_{\phi,\psi}. \tag{2.16}$$

Moreover, the distribution defined through

$$\langle \Theta_p \cdot \mathbf{n}, \eta \rangle = \int_\Omega \Theta_p \nabla\eta \mathrm{d}x - \int_\Omega \eta\rho\mathrm{d}x, \ \ \eta \in \mathcal{D}(\mathbb{R}^N) \tag{2.17}$$

is a Radon measure concentrated on $\partial\Omega$ which satisfies

$$\int_\Omega \Theta_p \cdot \nabla\eta\mathrm{d}x = \int_\Omega \eta\rho\mathrm{d}x + \int_{\partial\Omega} \eta\mathrm{d}(\Theta_p \cdot \mathbf{n}) \ \ \textit{for all } \eta \in W^{1,p}(\Omega), \tag{2.18}$$

and

$$\mathrm{supp}((\Theta_p \cdot \mathbf{n})^+) \subset \{u_p = \phi\} \ \ \textit{and} \ \ \mathrm{supp}((\Theta_p \cdot \mathbf{n})^-) \subset \{u_p = \psi\}. \tag{2.19}$$

Proof. We consider the following minimization problem of Finsler p-Laplace type:

$$\min_{u\in\mathcal{W}_{\phi,\psi}} \mathcal{F}_p(u) := \int_\Omega \frac{H^*(x,\nabla u)^p}{p}\mathrm{d}x - \int_\Omega u\rho \mathrm{d}x. \tag{2.20}$$

Observe that $\mathcal{W}_{\phi,\psi}$ is a closed, convex subset of $W^{1,p}(\Omega)$. The functional $\mathcal{F}_p$ is coercive, strictly convex, and lower semicontinuous on $\mathcal{W}_{\phi,\psi}$. Therefore $\mathcal{F}_p$ admits a unique minimizer on $\mathcal{W}_{\phi,\psi}$ which satisfies (2.16).

Now, to prove (2.18) we follow the main ideas of [23, Thereom 3.4]. Clearly, (2.16) implies $-\operatorname{div}(\Theta_p) = \rho$ in $\mathcal{D}^{'}(\Omega)$. It follows that the $\Theta_p\cdot\mathbf{n}$ defined by (2.17) is a distribution supported on $\partial\Omega$. Let us show moreover that

$$\operatorname{supp}(\Theta_p\cdot\mathbf{n}) \subset \{x\in\partial\Omega:\ u_p(x)=\phi(x)\}\cup\{x\in\partial\Omega:\ u_p(x)=\psi(x)\}.$$

Take a test function $\eta\in C^\infty(\overline{\Omega})$ whose support is disjoint from $\{x\in\partial\Omega:\ u_p(x)=\phi(x)\}\cup\{x\in\partial\Omega:\ u_p(x)=\psi(x)\}$. There exists some $\epsilon>0$ so that $u_p+t\eta$ remains admissible for (2.20) for $|t|<\epsilon$, i.e., $\phi\le u_p+t\eta\le\psi$. By optimality of u_p, we get the variational inequality

$$\int_\Omega \Theta_p\cdot\nabla(v-u_p)\mathrm{d}x \ge \int_\Omega (v-u_p)\rho\mathrm{d}x \ \text{ for all } v\in\mathcal{W}_{\phi,\psi}.$$

In particular, for $v=u_p+t\eta$, we get

$$t\int_\Omega \Theta_p\cdot\nabla\eta\mathrm{d}x \ge t\int_\Omega \eta\rho\mathrm{d}x.$$

This holds for positive and negative t such that $|t|\le\epsilon$. Consequently

$$\int_\Omega \Theta_p\cdot\nabla\eta\mathrm{d}x = \int_\Omega \eta\rho\mathrm{d}x.$$

In other words, $\langle\Theta_p\cdot\mathbf{n},\eta\rangle = 0$ and $\operatorname{supp}(\Theta_p\cdot\mathbf{n})\subset\{u_p=\phi\}\cup\{u_p=\psi\}$. We are now in a position to show that $\Theta_p\cdot\mathbf{n}$ is actually a Radon measure. Indeed, the inequality (2.15) implies that the two compact sets $\{x\in\partial\Omega:\ u_p(x)=\phi(x)\}$ and $\{x\in\partial\Omega:\ u_p(x)=\psi(x)\}$ are disjoint. There exist $\eta_1,\eta_2\in\mathcal{D}(\mathbb{R}^N)$ such that

$$\eta_1(x)=\begin{cases}1 \text{ on } \{u_p=\phi\},\\ 0 \text{ on } \{u_p=\psi\}\end{cases} \quad \text{and} \quad \eta_2(x)=\begin{cases}1 \text{ on } \{u_p=\psi\},\\ 0 \text{ on } \{u_p=\phi\}.\end{cases}$$

That we can write $\Theta_p\cdot\mathbf{n}=D_1+D_2$, where D_1,D_2 are distributions given by

$$\langle D_1,\eta\rangle = \langle\Theta_p\cdot\mathbf{n},\eta\eta_1\rangle \ \text{ and } \ \langle D_2,\eta\rangle = \langle\Theta_p\cdot\mathbf{n},\eta\eta_2\rangle.$$

That being said, for any positive test function η, we have that $\operatorname{supp}(\eta\eta_1)\cap\{u_p=\psi\}=\emptyset$, and for $0\le t<\epsilon$ we have $u_p+t(\eta\eta_1)\in\mathcal{W}_{\phi,\psi}$. Consequently

$$t\int_\Omega \Theta_p\cdot\nabla(\eta\eta_1)\mathrm{d}x \ge t\int_\Omega (\eta\eta_1)\rho\mathrm{d}x,$$

i.e.,

$$\langle D_1,\eta\rangle \ge 0. \tag{2.21}$$

On the other hand, for any positive test function η, we have that $\mathrm{supp}(\eta\eta_2) \cap \{u_p = \phi\} = \emptyset$, and for $-\epsilon < t \le 0$, we have that $u_p + t(\eta\eta_2) \in \mathcal{W}_{\phi,\psi}$. Consequently

$$t\int_\Omega \Theta_p \cdot \nabla(\eta\eta_2)\mathrm{d}x \ge t\int_\Omega (\eta\eta_2)\rho\mathrm{d}x.$$

In other words,

$$\langle D_2, \eta\rangle \le 0. \tag{2.22}$$

□

In conclusion, D_1 and $-D_2$ are positive distributions. Hence, they are positive Radon measures. It follows that the distribution $\Theta_p \cdot \mathbf{n}$ is a Radon measure on $\partial\Omega$. Moreover, (2.21) and (2.22) give (2.19).

Thanks to the proof of Proposition 2.3, we have the following description of the solution.

Corollary 2.4. *If $H^*(x,.) \in C^1(\mathbb{R}^N \setminus \{0\})$, then u_p is the unique weak solution of the problem*

$$\begin{cases} -\operatorname{div}(H^*(x,\nabla u_p)^{p-1}\partial_\xi H^*(x,\nabla u_p)) = \rho & \text{in } \Omega,\\ H^*(x,\nabla u_p)^{p-1}\partial_\xi H^*(x,\nabla u_p)\cdot \mathbf{n} \ge 0 & \text{on } \{u_p = \phi\},\\ H^*(x,\nabla u_p)^{p-1}\partial_\xi H^*(x,\nabla u_p)\cdot \mathbf{n} \le 0 & \text{on } \{u_p = \psi\},\\ H^*(x,\nabla u_p)^{p-1}\partial_\xi H^*(x,\nabla u_p)\cdot \mathbf{n} = 0 & \text{in } \{\phi < u_p < \psi\},\\ \phi \le u_p \le \psi & \text{on } \partial\Omega, \end{cases} \tag{2.23}$$

where $\mathbf{n}$ is the exterior normal to the boundary $\partial\Omega$ in the sense that $u_p \in \mathcal{W}_{\phi,\psi}$, $\Theta_p \in L^{p'}(\Omega)^N$, $\Theta_p \cdot \mathbf{n} \in \mathcal{M}_b(\partial\Omega)$, and the triplet $(u_p, \Theta_p, \Theta_p\cdot\mathbf{n})$ satisfies (2.18)–(2.19).

Remark 2.5. In order to simplify the presentation we have assumed that $H^*(x,.) \in C^1(\mathbb{R}^n \setminus \{0\})$. However, we do believe that all the results of this paper remain true without this assumption and that one needs just to replace the derivative of H^* with respect to the second variable by the subdifferential.

Remark 2.6. It is possible to use the same techniques for the HJ equation with double obstacles in the whole domain Ω. Namely, consider the following equation:

$$\begin{cases} F(x,\nabla u) = 0 & \text{in } [\phi \le u \le \psi],\\ u = g & \text{on } \partial\Omega, \end{cases} \tag{2.24}$$

where $\phi, \psi \in C(\overline{\Omega})$ such that $\phi \le \psi$ and $g \in C(\partial\Omega)$ is some compatible boundary data. Then, arguing as in Proposition 2.1, the maximal viscosity subsolution of (2.24) can be recovered by the following maximization problem:

$$\max_{z\in W_g^{1,\infty}(\Omega)} \left\{ \int_\Omega z(x)\mathrm{d}x,\ \sigma^*(x,\nabla z(x)) \le 1 \text{ and } \phi \le z \le \psi \text{ in } \Omega \right\}.$$

Moreover, the solution of (2.24) can be obtained by minimizing the functional $\mathcal{F}_p$ in (2.20) over the set $\{u \in W_g^{1,p}(\Omega) :\ \phi \le u \le \psi \text{ in } \Omega\}$.

3. Limits of Finsler p-Laplacian as $p \to \infty$. The strategy is to obtain some uniform bounds in p of ∇u_p; then we show that the triplet $(u_p, \Theta_p, \Theta_p\cdot\mathbf{n})$ converges (up to a subsequence) to optimal solutions of the corresponding Kantorovich–Rubinstein and Beckmann-type problems. The following result gathers main estimates that we will need later.

PROPOSITION 3.1 (main estimates). *Assume* (2.13) *is strict, that is,*

$$\phi(x) - \psi(y) < d_H(y,x) \text{ for all } x, y \in \partial\Omega.$$

Then, we have

(i) *estimate on* u_p

$$|u_p(x) - u_p(y)| \le C|x-y|^r, \quad \text{for all} \quad x, y \in \Omega; \tag{3.1}$$

(ii) *estimates on* $\Theta_p \cdot \mathbf{n}$

$$\int_{\partial\Omega} \mathrm{d}(\Theta_p \cdot \mathbf{n})^+ \le C_1 \quad \text{and} \quad \int_{\partial\Omega} \mathrm{d}(\Theta_p \cdot \mathbf{n})^- \le C_2; \tag{3.2}$$

(iii) *estimate on* Θ_p

$$\int_\Omega |\Theta_p| \mathrm{d}x \le C, \tag{3.3}$$

where r, C, C_1, C_2 *are positive constants independent from* p.

Proof. First, we prove (i). Define $v(x) = \min_{y\in\partial\Omega} \psi(y) + d_H(y,x)$. Regarding the compatibility condition (2.13), we have $\phi \le v \le \psi$ on $\partial\Omega$. It is not difficult to see that v is $1-$Lipschitz with respect to d_H, and equivalently (see, e.g., [12, Proposition 2.1]), we have that $H^*(x, \nabla v(x)) \le 1$ a.e. in Ω. Using the fact that u_p is a minimizer of $\mathcal{F}_p$, we have

$$\int_\Omega \frac{H^*(x,\nabla u_p)^p}{p} \mathrm{d}x - \int_\Omega u_p \rho \mathrm{d}x \le \int_\Omega \frac{H^*(x,\nabla v)^p}{p} \mathrm{d}x - \int_\Omega v\rho \mathrm{d}x \le \frac{|\Omega|}{p} - \int_\Omega v\rho \mathrm{d}x. \tag{3.4}$$

Thanks to Theorem 2.E in [28], there is a Morrey-type inequality independent of p

$$\|u\|_{L^\infty(\Omega)} \le C_\Omega \|\nabla u\|_{L^p(\Omega)} \text{ for any } u \in W_0^{1,p}(\Omega), \; p > N+1,$$

where the constant C_Ω does not depend on p and u. Observing that we can apply the above inequality to $(u_p - \max_{\partial\Omega}\psi)^+$ and $(u_p - \min_{\partial\Omega}\phi)^-$ which are in $W_0^{1,p}(\Omega)$ to obtain

$$\|u_p^+\|_{L^\infty(\Omega)} \le C_\Omega \|\nabla u_p\|_{L^p(\Omega)} + |\max_{\partial\Omega} \psi|$$

and

$$\|u_p^-\|_{L^\infty(\Omega)} \le C_\Omega \|\nabla u_p\|_{L^p(\Omega)} + |\min_{\partial\Omega} \phi|.$$

So

$$\|u_p\|_{L^\infty(\Omega)} \le C_1 \|\nabla u_p\|_{L^p(\Omega)} + C_2.$$

From (3.4) and the preceding inequality we deduce that

$$\int_\Omega \frac{H^*(x,\nabla u_p)^p}{p} \mathrm{d}x \le \frac{|\Omega|}{p} - \int_\Omega v\rho \mathrm{d}x + \int_\Omega u_p \rho \mathrm{d}x \le C_3(1 + \|\nabla u_p\|_{L^p(\Omega)}),$$

where C_3 is a positive constant not depending on p. Combining this with (2.5), we get

$$\|H^*(x,\nabla u_p)\|^p_{L^p(\Omega)} \leq C_4 p(1+\|H^*(x,\nabla u_p)\|_{L^p(\Omega)})$$

which implies that

$$\|H^*(x,\nabla u_p)\|_{L^p(\Omega)} \leq (C_5 p)^{\frac{1}{p-1}} \tag{3.5}$$

for some constant C_5 independent from p. Again, by (2.5), we get

$$\|\nabla u_p\|_{L^p(\Omega)} \leq C_6. \tag{3.6}$$

Now take some $N < m \leq p$. Then by Hölder's inequality

$$\|\nabla u_p\|_{L^m(\Omega)} \leq |\Omega|^{\frac{p-m}{pm}} \|\nabla u_p\|_{L^p(\Omega)}. \tag{3.7}$$

Thanks to (3.6), (3.7), and the Morrey–Sobolev embedding from $W^{1,m}(\Omega)$ to Hölder spaces,

$$|u_p(x) - u_p(y)| \leq C_7 |x-y|^{1-\alpha}$$

with $\alpha = \frac{N}{m}$.

Now, let us prove (ii). We consider as before $v(x) = \min_{y\in\partial\Omega} \psi(y) + d_H(y,x)$. We have

$$\int_{\partial\Omega} (u_p - v)\mathrm{d}(\Theta_p\cdot\mathbf{n}) = \int_\Omega \Theta_p\cdot\nabla(u_p - v)\mathrm{d}x - \int_\Omega (u_p - v)\rho\mathrm{d}x.$$

In other words

$$\int_\Omega (u_p - v)\rho\mathrm{d}x = \int_\Omega \Theta_p\cdot\nabla(u_p - v)\mathrm{d}x + \int_{\{u_p=\psi\}} (\psi - v)\mathrm{d}(\Theta_p\cdot\mathbf{n})^- - \int_{\{u_p=\phi\}} (\phi - v)\mathrm{d}(\Theta_p\cdot\mathbf{n})^+.$$

We see that $\phi < v \leq \psi$ on $\partial\Omega$ so that $\psi - v \geq 0$ and $\phi - v < 0$; thus $\phi - v < -C_1$ for some positive constant C_1. So we obtain

$$\int_\Omega \Theta_p\cdot\nabla u_p \mathrm{d}x + C_1\int_{\partial\Omega} \mathrm{d}(\Theta_p\cdot\mathbf{n})^+ \leq \int_\Omega (u_p - v)\rho\mathrm{d}x + \int_\Omega \Theta_p\cdot\nabla v\mathrm{d}x. \tag{3.8}$$

Since H^* is a Finsler metric, we have by Euler's homogeneous function theorem (see, e.g., [24]) that $\partial_\xi H^*(x,\xi)\cdot\xi = H^*(x,\xi)$ for any $\xi\in\mathbb{R}^N$. Thus

$$\int_\Omega \Theta_p\cdot\nabla u_p\mathrm{d}x = \int_\Omega H^*(x,\nabla u_p)^{p-1}\partial_\xi H^*(x,\nabla u_p)\cdot\nabla u_p\mathrm{d}x = \int_\Omega H^*(x,\nabla u_p)^p\mathrm{d}x.$$

Using this fact in (3.8), we get

$$\int_\Omega H^*(x,\nabla u_p)^p\mathrm{d}x + C_1\int_{\partial\Omega}\mathrm{d}(\Theta_p\cdot\mathbf{n})^+ \leq C_2 + \int_\Omega \Theta_p\cdot\nabla v\mathrm{d}x,$$

where $C_2 > 0$ is independent from p. On the other hand, thanks to (2.7) we have

$$\begin{aligned}\int_\Omega \Theta_p \cdot \nabla v \mathrm{d}x &\le \int_\Omega H(x,\Theta_p) H^*(x,\nabla v)\mathrm{d}x \\ &= \int_\Omega H(x, H^*(x,\nabla u_p)^{p-1}\partial_\xi H^*(x,\nabla u_p))H^*(x,\nabla v)\mathrm{d}x \\ &= \int_\Omega H^*(x,\nabla u_p)^{p-1} H(x,\partial_\xi H^*(x,\nabla u_p))H^*(x,\nabla v)\mathrm{d}x \\ &= \int_\Omega H^*(x,\nabla u_p)^{p-1}H^*(x,\nabla v)\mathrm{d}x,\end{aligned}$$

where we have used the homogeneity of H and (2.10). Using Hölder and Young's inequalities and the fact that $H^*(x,\nabla v) \le 1$ a.e., we get

$$\begin{aligned}\int_\Omega H^*(x,\nabla u_p)^{p-1}H^*(x,\nabla v)\mathrm{d}x &\le \left(\int_\Omega H^*(x,\nabla u_p)^{(p-1)p'}\mathrm{d}x\right)^{\frac{1}{p'}} |\Omega|^{\frac{1}{p}} \\ &\le \frac{p-1}{p}\int_\Omega H^*(x,\nabla u_p)^p \mathrm{d}x + \frac{1}{p}|\Omega|.\end{aligned}$$

We deduce that

$$\frac{1}{p}\int_\Omega H^*(x,\nabla u_p)^p\mathrm{d}x + C_1\int_{\partial\Omega}\mathrm{d}(\Theta_p\cdot\mathbf{n})^+ \le C_2 + \frac{1}{p}|\Omega|.$$

Therefore

$$\int_{\partial\Omega}\mathrm{d}(\Theta_p\cdot\mathbf{n})^+ \le C_3 \tag{3.9}$$

for some positive constant C_3 independent of p. Set $w(x) = \max_{y\in\partial\Omega}\phi(y) - d_H(y,x)$. Observe that $\phi \le w < \psi$, and following the same lines we get that

$$\int_{\partial\Omega}\mathrm{d}(\Theta_p\cdot\mathbf{n})^- \le C_4. \tag{3.10}$$

As for Θ_p, we have

$$\int_\Omega H^*(x,\nabla u_p)^p\mathrm{d}x = \int_\Omega \Theta_p\cdot\nabla u_p\mathrm{d}x = \int_{\partial\Omega} u_p\mathrm{d}(\Theta_p\cdot\mathbf{n}) + \int_\Omega u_p\rho\mathrm{d}x.$$

Keeping in mind (3.9) and (3.10), Hölder's inequality gives

$$\int_\Omega H^*(x,\nabla u_p)^{p-1}\mathrm{d}x \le C_5;$$

this proves (iii). ∎

Thanks to Proposition 3.1, we can state the main result.

THEOREM 3.2. *Let u_p be a minimizer of $\mathcal{F}_p$. Then, up to a subsequence, $u_p \rightrightarrows \mathbf{u}$ on $\overline{\Omega}$, where $\mathbf{u}$ solves the following variant of Kantorovich–Rubinstein problem:*

$$(\mathcal{KR})_H: \ \max\left\{\int_\Omega u\mathrm{d}\rho: \ H^*(x,\nabla u)\le 1 \ a.e., \ \phi\le u\le\psi \ on\ \partial\Omega\right\}.$$

Moreover, there exists a couple $(\Theta,\theta)\in\mathcal{M}_b(\Omega)^N\times\mathcal{M}_b(\partial\Omega)$ such that there is the following:

(i) *Up to a subsequence*

$$(\Theta_p, \Theta_p \cdot \mathbf{n}) \rightharpoonup (\Theta, \theta) \quad \text{in } \mathcal{M}_b(\Omega)^N \times \mathcal{M}_b(\partial\Omega) - \text{weak}^*.$$

(ii) (Θ, θ) *solves the Beckmann problem*

$$(\mathcal{B})_H : \min_{\substack{\Phi \in \mathcal{M}_b(\Omega)^N \\ \nu \in \mathcal{M}_b(\partial\Omega)}} \left\{ \int_\Omega H(x, \frac{\Phi}{|\Phi|}) \mathrm{d}|\Phi| + \int_{\partial\Omega} \psi \mathrm{d}\nu^- - \int_{\partial\Omega} \phi \mathrm{d}\nu^+ : \quad -\mathrm{div}(\Phi) = \rho + \nu \text{ in } \mathcal{D}'(\mathbb{R}^N) \right\}.$$

(iii) *The couple* $(\mathbf{u}, \Theta)$ *solves the PDE*

$$\left\{ \begin{array}{ll} -\mathrm{div}(\Theta) = \rho & \text{in } \Omega, \\ \Theta(x) \cdot \nabla \mathbf{u}(x) = H(x, \Theta) & \text{in } \Omega, \\ \phi \leq \mathbf{u} \leq \psi & \text{on } \partial\Omega, \end{array} \right. \tag{3.11}$$

in the following sense: $(\mathbf{u}, \Theta) \in \mathcal{W}_{\phi,\psi} \times \mathcal{M}_b(\Omega)^N$, $\Theta \cdot \mathbf{n} = \theta \in \mathcal{M}_b(\partial\Omega)$,

$$\frac{\Theta}{|\Theta|} \cdot \nabla_{|\Theta|} \mathbf{u} = H\left(., \frac{\Theta}{|\Theta|}\right), \quad |\Theta| - \text{a.e. in } \Omega, \tag{3.12}$$

$$\mathrm{supp}(\theta^+) \subset \{\mathbf{u} = \phi\} \quad \text{and} \quad \mathrm{supp}(\theta^-) \subset \{\mathbf{u} = \psi\},$$

and

$$\int_\Omega \Theta \cdot \nabla \eta \, \mathrm{d}x = \int_\Omega \eta \rho \, \mathrm{d}x + \int_{\partial\Omega} \eta \, \mathrm{d}\theta \quad \text{for all } \eta \in W^{1,\infty}(\Omega).$$

Proof. **The case where the inequality (2.13) is strict.** First, we see that, thanks to (3.1), we have by Ascoli–Arzelà's theorem, up to a subsequence, $u_p \rightrightarrows \mathbf{u}$ on $\overline{\Omega}$ for some continuous function $\mathbf{u}$ satisfying $\phi \leq \mathbf{u} \leq \psi$ on $\partial\Omega$. It is clear that $\mathbf{u} \in W^{1,\infty}(\Omega)$.

We are now in a position to show that $\mathbf{u}$ solves $(\mathcal{KR})_H$. To do so, we take any $v \in \mathcal{W}_{\phi,\psi}$ such that $H^*(x, \nabla v(x)) \leq 1$ a.e.. Using the optimality of u_p we see that

$$-\int_\Omega u_p \rho \mathrm{d}x \leq \mathcal{F}_p(u_p) \leq \mathcal{F}_p(v) \leq \frac{|\Omega|}{p} - \int_\Omega v \rho \mathrm{d}x.$$

Taking the limit up to a subsequence, we get

$$\sup \left\{ \int_\Omega v \rho \mathrm{d}x : \ H^*(x, \nabla v) \leq 1 \text{ a.e., } \phi \leq v \leq \psi \text{ on } \partial\Omega \right\} \leq \int_\Omega \mathbf{u} \rho \mathrm{d}x.$$

It remains to show that $\mathbf{u}$ is $1-$Lipschitz with respect to d_H, that is, $H^*(x, \nabla \mathbf{u}(x)) \leq 1$ a.e.. Recall that $\phi \leq \mathbf{u} \leq \psi$ on $\partial\Omega$. Again, using (3.5), we consider $N < m \leq p$, and we use Hölder's inequality to get

$$\|H^*(x, \nabla u_p)\|_{L^m(\Omega)} \leq (C_5 p)^{\frac{1}{p-1}} |\Omega|^{\frac{p-m}{pm}}.$$

Since $u_p \rightrightarrows \mathbf{u}$ uniformly in $\overline{\Omega}$, we can assume that up to a subsequence $u_p \rightharpoonup \mathbf{u}$ weakly in $W^{1,m}(\Omega)$, and particularly, $\nabla u_p \rightharpoonup \nabla \mathbf{u}$ weakly in $L^m(\Omega, \mathbb{R}^N)$. Mazur's lemma (see

[11] for an example) ensures the existence of a convex combination of ∇u_{p_k} converging in norm toward $\nabla \mathbf{u}$. More precisely, there exists $\{U_i\}$ such that

$$U_i = \sum_{k=i}^{n_i} \alpha_k^i \nabla u_{p_k},$$

where $\sum_{k=i}^{n_i} \alpha_i^k = 1$ and $\alpha_k^i \geq 0,\ i \leq k \leq n_i$, and $\|U_i - \nabla \mathbf{u}\|_{L^m(\Omega)} \to 0$ as $i \to +\infty$. Since H^* is continuous, we have

$$\begin{aligned} \|H^*(x, \nabla \mathbf{u})\|_{L^m(\Omega)} &\leq \liminf_{i\to\infty} \|H^*(x, \sum_{k=i}^{n_i} \alpha_k^i \nabla u_{p_k})\|_{L^m(\Omega)} \\ &\leq \liminf_{i\to\infty} \sum_{k=i}^{n_i} \alpha_k^i \|H^*(x, \nabla u_{p_k})\|_{L^m(\Omega)} \\ &\leq \liminf_{i\to\infty} \sum_{k=i}^{n_i} \alpha_k^i (C_5 p_k)^{\frac{1}{p_k-1}} |\Omega|^{\frac{p_k-m}{mp_k}} = |\Omega|^{\frac{1}{m}}. \end{aligned}$$

Taking $m \to \infty$, we get $H^*(x, \nabla u(x)) \leq 1$ a.e. $x \in \Omega$. On the other hand, we see that (3.3) and (3.2) imply that Θ_p and $\Theta_p \cdot \mathbf{n}$ are bounded in $\mathcal{M}_b(\overline{\Omega})$ and $\mathcal{M}_b(\partial\Omega)$, respectively. As a consequence, there exist $\Theta \in \mathcal{M}_b(\overline{\Omega})^N$ and $\theta \in \mathcal{M}_b(\partial\Omega)$ such that up to a subsequence

$$\Theta_p \rightharpoonup \Theta \text{ weakly* as } p \to \infty$$

and

$$\Theta_p \cdot \mathbf{n} \rightharpoonup \theta \text{ weakly* as } p \to \infty.$$

Next, take any admissible potential $v \in C^1(\Omega)$ for $(\mathcal{KR})_H$ and an admissible couple of flows $(\Psi, \nu) \in \mathcal{M}_b(\Omega)^N \times \mathcal{M}_b(\partial\Omega)$ for $(\mathcal{B})_H$. Since $H^*(x, \nabla v) \leq 1$ for a.e. $x \in \Omega$, we have

$$\begin{aligned} \int_\Omega H\left(x, \frac{\Psi}{|\Psi|}\right) \mathrm{d}|\Psi| &\geq \int_\Omega H\left(x, \frac{\Psi}{|\Psi|}\right) H^*(x, \nabla v) \mathrm{d}|\Psi| \\ &\geq \int_\Omega \frac{\Psi}{|\Psi|} \nabla v \mathrm{d}|\Psi| \\ &\geq \int_\Omega v \mathrm{d}\rho + \int_{\partial\Omega} \phi \mathrm{d}\nu^+ - \int_{\partial\Omega} \psi \mathrm{d}\nu^- \end{aligned}$$

and consequently

$$\int_\Omega H\left(x, \frac{\Psi}{|\Psi|}\right) \mathrm{d}|\Psi| + \int_{\partial\Omega} \psi \mathrm{d}\nu^- - \int_{\partial\Omega} \phi \mathrm{d}\nu^+ \geq \int_\Omega v \mathrm{d}\rho.$$

In particular, this implies that

$$\min(\mathcal{B})_H \geq \max(\mathcal{KR})_H.$$

On the other hand, using Hölder's inequality combined with (2.8)–(2.9), we get

$$\begin{aligned}\int_\Omega H\left(x,\frac{\Theta}{|\Theta|}\right)\mathrm{d}|\Theta| &\le \liminf_p \int_\Omega H\left(x, H^*(x,\nabla u_p)^{p-1}\partial_\xi H^*(x,\nabla u_p)\right)\mathrm{d}x\\ &= \liminf_p \int_\Omega H^*(x,\nabla u_p)^{p-1} H(x,\partial_\xi H^*(x,\nabla u_p))\mathrm{d}x\\ &\le \liminf_p \left(\int_\Omega H^*(x,\nabla u_p)^p \mathrm{d}x\right)^{\frac{p-1}{p}}\\ &= \liminf_p \left(\int_\Omega H^*(x,\nabla u_p)^{p-1}\ \partial_\xi H^*(x,\nabla u_p)\cdot\nabla u_p \mathrm{d}x\right)^{\frac{p-1}{p}}\\ &= \liminf_p \left(\int_\Omega \nabla u_p \mathrm{d}\Theta_p\right)^{\frac{p-1}{p}}\\ &= \liminf_p \left(\int_\Omega u_p\rho\mathrm{d}x + \int_{\partial\Omega} u_p \mathrm{d}(\Theta_p\cdot\mathbf{n})\right)^{\frac{p-1}{p}}\\ &= \int_\Omega \mathbf{u}\rho\mathrm{d}x + \int_{\partial\Omega}\phi\mathrm{d}\theta^+ - \int_{\partial\Omega}\psi\mathrm{d}\theta^-.\end{aligned}$$

This implies that

$$\min(\mathcal{B})_H \ \le \int_\Omega H\left(x,\frac{\Theta}{|\Theta|}\right)\mathrm{d}|\Theta| - \int_{\partial\Omega}\phi\mathrm{d}\theta^+ + \int_{\partial\Omega}\psi\mathrm{d}\theta^- \le \int_\Omega \mathbf{u}\rho\mathrm{d}x = \max(\mathcal{KR})_H.$$

Thus

$$\min(\mathcal{B})_H \ = \int_\Omega H\left(x,\frac{\Theta}{|\Theta|}\right)\mathrm{d}|\Theta| - \int_{\partial\Omega}\phi\mathrm{d}\theta^+ + \int_{\partial\Omega}\psi\mathrm{d}\theta^- = \int_\Omega \mathbf{u}\rho\mathrm{d}x = \max(\mathcal{KR})_H,$$

which implies the optimality of $\mathbf{u}$ and (Φ,θ).

Now it remains to show the results for the general case where the inequality (2.13) needs not to be strict.

We proceed by approximations. Consider two sequences $\{\phi_n\}_n$ and $\{\psi_n\}_n$ of continuous functions on $\partial\Omega$ such that

$$\phi_n(x) - \psi_n(y) < d_H(y,x) \text{ for all } x,y\in\partial\Omega,$$

and

$$\phi_n \rightrightarrows \phi \ \text{ and } \ \psi_n \rightrightarrows \psi \text{ on } \partial\Omega.$$

Then, thanks to the previous case, there exists a sequence of $\{\mathbf{u}_n\}_n \in \mathcal{W}_{\phi_n,\psi_n}$ such that $H^*(x,\nabla\mathbf{u}_n)\le 1$ a.e Ω. In addition, consider the corresponding solutions to the Beckmann problem (Θ_n,θ_n). We then have

$$\int_\Omega \mathbf{u}_n\mathrm{d}\rho = \int_\Omega H\left(x,\frac{\Theta_n}{|\Theta_n|}\right)\mathrm{d}|\Theta_n| - \int_{\partial\Omega}\phi_n\mathrm{d}\theta_n^+ + \int_{\partial\Omega}\psi_n\mathrm{d}\theta_n^- = \min(\mathcal{B})_H. \tag{3.13}$$

Then we deduce by the previous arguments that

$$\mathbf{u}_n \rightrightarrows \mathbf{u} \text{ uniformly in } \overline{\Omega} \text{ with } \ H^*(x,\nabla\mathbf{u})\le 1 \text{ a.e. and } \ \phi\le\mathbf{u}\le\psi \text{ in } \partial\Omega.$$

Next, we follow the main ideas of the proof of Proposition 3.1. Define $v_n(x) = \min_{y\in\partial\Omega}\{\psi_n(y) + d_H(y,x)\}$. Then

$$\int_\Omega \Theta_n \cdot \nabla \mathbf{u}_n \mathrm{d}x + C_1 \int_{\partial\Omega} \mathrm{d}\theta_n^+ \leq \int_\Omega (\mathbf{u}_n - v_n)\rho \mathrm{d}x + \int_\Omega \Theta_n \cdot \nabla v_n \mathrm{d}x, \tag{3.14}$$

where C_1 is a positive constant independent from n. Using (3.11), we have

$$\int_\Omega \Theta_n \cdot \nabla \mathbf{u}_n \mathrm{d}x = \int_\Omega H\left(x, \frac{\Theta_n}{|\Theta_n|}\right) \mathrm{d}|\Theta_n|.$$

On the other hand, since $H^*(x, \nabla v_n(x)) \leq 1$ a.e, we get

$$\int_\Omega \Theta_n \cdot \nabla v_n \mathrm{d}x \leq \int_\Omega H(x,\Theta_n)H^*(x,\nabla v_n)\mathrm{d}x \leq \int_\Omega H\left(x, \frac{\Theta_n}{|\Theta_n|}\right) \mathrm{d}|\Theta_n|.$$

Combining these facts in (3.14) and using (2.5) we get

$$\int_{\partial\Omega} \mathrm{d}\theta_n^+ \leq C, \text{ with } C > 0. \tag{3.15}$$

Similarly, working with $w_n(x) = \max_{y\in\partial\Omega} \phi_n(y) - d_H(y,x)$ instead of v_n, we get

$$\int_{\partial\Omega} \mathrm{d}\theta_n^- \leq C, \text{ with } C > 0. \tag{3.16}$$

As for Θ_n, we deduce from (2.5), (3.13), (3.15), and (3.16) that

$$\int_\Omega |\Theta_n| \mathrm{d}x \leq C.$$

Then, up to a subsequence, $(\Theta_n, \theta_n) \rightharpoonup (\Theta, \theta)$ weakly* as $n \to \infty$. Thus, passing to the limit in (3.13), the proof is complete.

Finally, for the proof of the last item (iii), by passing to the limit, we recover the conditions

$$\operatorname{supp}(\theta^+) \subset \{\mathbf{u} = \phi\} \quad \text{and} \quad \operatorname{supp}(\theta^-) \subset \{\mathbf{u} = \psi\},$$

and

$$\int_\Omega \Theta \cdot \nabla \eta \, \mathrm{d}x = \int_\Omega \eta\rho \, \mathrm{d}x + \int_{\partial\Omega} \eta \, \mathrm{d}\theta \text{ for all } \eta \in W^{1,\infty}(\Omega).$$

The equation

$$\frac{\Theta}{|\Theta|} \cdot \nabla_{|\Theta|} \mathbf{u} = H\left(., \frac{\Theta}{|\Theta|}\right), \quad |\Theta| - \text{a.e. in } \Omega$$

is due to the optimality of $\mathbf{u}$ and Φ (see, for example, [20, 25]). ◻

By uniqueness of the maximal viscosity subsolution of (2.1) we easily deduce the following corollary.

Corollary 3.3. *Let $H = \sigma$, with σ being the support function of the 0-sublevel sets of the Hamiltonian F in* (2.1). *Then the whole sequence $\{\mathbf{u}_p\}_p$ converges uniformly to the solution $\mathbf{u}$ of* (2.1).

Now let us state the PDE satisfied by the potential $\mathbf{u}$ and the flow Θ, which in particular will give a characterization of the HJ equation (2.1).

PROPOSITION 3.4. *The couple* $(\mathbf{u}, \Theta)$ *given by Theorem* 3.2 *is a solution of the PDE*

$$\begin{cases} -\operatorname{div}(\Theta) = \rho & \text{in } \Omega, \\ \Theta \in \partial \mathbb{I}_{B_{H^*(x,.)}}(\nabla \mathbf{u}) & \text{in } \Omega, \\ \phi \leq \mathbf{u} \leq \psi & \text{on } \partial\Omega \end{cases}$$

in the sense that $(\mathbf{u}, \Theta) \in \mathcal{W}_{\phi,\psi} \times \mathcal{M}_b(\Omega)^N, \Theta \cdot \mathbf{n} = \theta \in \mathcal{M}_b(\partial\Omega)$,

$$\Theta \in \partial \mathbb{I}_{B_{H^*(x,.)}}(\nabla_{|\Theta|}\mathbf{u}), \quad |\Theta| - a.e. \text{ in } \Omega,$$

$$\operatorname{supp}(\theta^+) \subset \{\mathbf{u} = \phi\} \quad \text{and} \quad \operatorname{supp}(\theta^-) \subset \{\mathbf{u} = \psi\},$$

and

$$\int_\Omega \Theta \cdot \nabla \eta \, \mathrm{d}x = \int_\Omega \eta \rho \, \mathrm{d}x + \int_{\partial\Omega} \eta \, \mathrm{d}\theta \quad \text{for all } \eta \in W^{1,\infty}(\Omega).$$

In particular, taking $H = \sigma$, *with* σ *being the support function of the* 0*-sublevel sets of the Hamiltonian* F, *the maximal viscosity subsolution* $\mathbf{u}$ *of* (2.1) *is uniquely characterized by the existence of* $\Theta \in \mathcal{M}_b(\Omega)^N$ *such that the couple* $(\mathbf{u}, \Theta)$ *is a solution of the PDE*

$$\begin{cases} -\operatorname{div}(\Theta) = 1 & \text{in } \Omega, \\ \Theta \in \partial \mathbb{I}_{Z(x)}(\nabla \mathbf{u}) & \text{in } \Omega, \\ \phi \leq \mathbf{u} \leq \psi & \text{on } \partial\Omega \end{cases}$$

Proof. The divergence and boundary constraints follow from Theorem 3.2, and

$$\Theta \in \partial \mathbb{I}_{B_{H^*(x,.)}}(\nabla_{|\Theta|}\mathbf{u})$$

is recovered by (3.12). ∎

Unlike in the Euclidean case $H = |\cdot|$, where the optimal flow Θ can be linked to the transport density and the gradient of the Kantorovich potential $\mathbf{u}$ (see (1.7)), dealing with a general Finsler metric H it is not straightforward how to phrase the flow Θ explicitly in such a way. The following result points out two particular situations showing how this is possible.

COROLLARY 3.5. *Let* $(\mathbf{u}, \Theta)$ *be a solution of the PDE* (3.11) *in the sense of Theorem* 3.2. *If*

$$|\Theta| \ll \mathcal{L}^N,$$

then, setting

$$\omega := H(x, \Theta), \tag{3.17}$$

we have

$$\Theta = \omega \, \partial_\xi H^*(x, \nabla \mathbf{u}) \quad \mathcal{L}^N - \ a.e. \ x \in \Omega$$

and

$$\omega \, (H^*(x, \nabla \mathbf{u}) - 1) = 0 \quad \mathcal{L}^N - \ a.e. \ x \in \Omega.$$

Proof. If $|\Theta| \ll \mathcal{L}^N$, then $\nabla_{|\Theta|}\mathbf{u} = \nabla\mathbf{u}, \mathcal{L}^N-$ a.e. in Ω, and by taking ω as in (3.17), the relationship (3.12) implies that $\Theta \cdot \nabla u = \omega\, \mathcal{L}^N-$ a.e. in Ω. Since moreover $H^*(x, \nabla\mathbf{u}) \leq 1$, then by definition of H^*, we get

$$\Theta = \omega\, \partial_\xi H^*(x, \nabla_\omega \mathbf{u}) \text{ and } \omega\,(H^*(., \nabla_\omega \mathbf{u}) - 1) = 0, \quad \mathcal{L}^N - \text{a.e. in } \Omega. \qquad \square$$

COROLLARY 3.6. *Let* $(\mathbf{u}, \Theta)$ *be a solution of* (3.11) *in the sense of Theorem* 3.2. *We set again*

$$\omega := H(x, \Theta),$$

and we assume moreover that

$$H^*(x, \nabla_\omega \mathbf{u}) \leq 1 \quad \omega - a.e.\ x \in \Omega. \tag{3.18}$$

Then

$$\Theta = \omega\, \partial_\xi H^*(x, \nabla_\omega \mathbf{u}),$$

and

$$H^*(x, \nabla_\omega \mathbf{u}) = 1 \quad \omega - a.e.\ x \in \Omega.$$

Proof. See that $\nabla_{|\Theta|}\mathbf{u} = \nabla_\omega \mathbf{u}$ and

$$H\left(x, \frac{\mathrm{d}\Theta}{\mathrm{d}\omega}\right) = 1 \quad \omega - \text{a.e. } \Omega.$$

So, in one hand, using the fact that

$$\nabla_{|\Theta|} u \cdot \frac{\Theta}{|\Theta|} = H\left(x, \frac{\Theta}{|\Theta|}\right) \quad |\Theta| - \text{a.e. } \Omega,$$

we have

$$\nabla_\omega \mathbf{u} \cdot \frac{\mathrm{d}\Theta}{\mathrm{d}\omega} = \nabla_{|\Theta|} u \cdot \frac{\mathrm{d}\Theta}{\mathrm{d}\omega} = 1 \quad \omega - \text{a.e. } \Omega.$$

On the other hand, we see that

$$\nabla_\omega u \cdot \frac{\mathrm{d}\Theta}{\mathrm{d}\omega} \leq H^*(x, \nabla_\omega u)\, H\left(x, \frac{\mathrm{d}\Theta}{\mathrm{d}\omega}\right) = H^*(x, \nabla_\omega u) \quad \omega - \text{a.e. } \Omega.$$

So, assuming (3.18), we get

$$1 = \nabla_\omega u \cdot \frac{\mathrm{d}\Theta}{\mathrm{d}\omega} = H^*(x, \nabla_\omega u)\, H\left(x, \frac{\mathrm{d}\Theta}{\mathrm{d}\omega}\right) = H^*(x, \nabla_\omega u) \quad \omega - \text{a.e. } \Omega. \qquad \square$$

Thus the results follow by definition of H^*.

Remark 3.7. Combining Theorem 3.2 and Corollaries 3.5–3.6, the couple $(\omega := H(x, \Theta), \mathbf{u})$ solves the associated Monge–Kantorovich system to $(\mathcal{KR})_H$ and $(\mathcal{B})_H$:

$$\begin{cases} -\operatorname{div}(\omega \partial_\xi H^*(x, \nabla_\omega \mathbf{u})) = \rho & \text{in } \Omega, \\ \partial_\xi H^*(x, \nabla_\omega \mathbf{u}) \cdot \mathbf{n} \geq 0 & \text{on } \{\mathbf{u} = \phi\}, \\ \partial_\xi H^*(x, \nabla_\omega \mathbf{u}) \cdot \mathbf{n} \leq 0 & \text{on } \{\mathbf{u} = \psi\}, \\ \partial_\xi H^*(x, \nabla_\omega u) \cdot \mathbf{n} = 0 & \text{in } \{\phi < \mathbf{u} < \psi\}, \\ \phi \leq \mathbf{u} \leq \psi & \text{on } \partial\Omega, \\ H^*(x, \nabla_\omega \mathbf{u}) \leq 1 & \text{in } \Omega, \\ H^*(x, \nabla_\omega \mathbf{u}) = 1 & \omega - \text{a.e.}. \end{cases} \tag{3.19}$$

In particular, given a positive continuous function $k : \overline{\Omega} \to \mathbb{R}$, and define the following Finsler metric $H(x,p) = k(x)|p|$ for $(x,p) \in \overline{\Omega} \times \mathbb{R}^N$. We easily see that its dual reads

$$H^*(x,q) = \frac{|q|}{k(x)},$$

and the systems (2.23)–(3.19) reduce the ones studied in [10].

Moreover, if the Finsler metric is defined via the so-called Minkowski functional (or gauge function)

$$\mathbf{g}_K(p) = \inf\{t > 0 : \ t^{-1}p \in K\},$$

where K is a convex, closed, and bounded set $\mathbb{R}^N$, then considering $H^*(x,p) = \mathbf{g}_K(p)$ and $\phi = \psi$, we recover the Monge–Kantorovich system studied in [8].

4. Connection with Monge–Kantorovich problem. Let us recall that we can derive a dual problem to $(\mathcal{KR})_H$ using perturbation techniques (as in [10, 12]) to get the following Kantorovich problem:

$$(\mathcal{K})_H : \quad \min_{\gamma \in \Pi(\rho^+,\rho^-)} \left\{ \int_{\overline{\Omega}\times\overline{\Omega}} d_H(x,y)\mathrm{d}\gamma(x,y) + \int_{\partial\Omega} \psi(y)\mathrm{d}(\pi_y)_\sharp\gamma - \int_{\partial\Omega} \phi(x)\mathrm{d}(\pi_x)_\sharp\gamma \right\}.$$

Here $\Pi(\rho^+,\rho^-) = \{\gamma \in \mathcal{M}^+(\overline{\Omega}\times\overline{\Omega}) : \ (\pi_x)_\sharp\gamma \llcorner \Omega = \rho^+, (\pi_y)_\sharp\gamma \llcorner \Omega = \rho^-\}$, with π_x and π_y standing for the usual projections of $\overline{\Omega}\times\overline{\Omega}$ onto $\overline{\Omega}$, that is, $\pi_x(x,y) = x$ and $\pi_y(x,y) = y$ for any $(x,y) \in \overline{\Omega}\times\overline{\Omega}$ and

$$(\pi_x)_\sharp\gamma \llcorner \Omega = \rho^+ \Leftrightarrow \gamma(A \times \overline{\Omega}) = \rho^+(A) \text{ for any Borelean } A \subset \Omega,$$

$$(\pi_y)_\sharp\gamma \llcorner \Omega = \rho^- \Leftrightarrow \gamma(\overline{\Omega} \times B) = \rho^-(B) \text{ for any Borelean } B \subset \Omega.$$

The existence of optimal solution to $(\mathcal{K})_H$ can be obtained using the direct method of calculus of variations. Moreover, all the extremal values coincide:

$$\min(\mathcal{B})_H = \min(\mathcal{K})_H = \max(\mathcal{KR})_H. \tag{4.1}$$

Here ϕ and ψ play the role of import/export costs for the Kantorovich problem $(\mathcal{K})$ as in [10, 23] for the Euclidean and Riemannian costs. In addition, we show that the measure θ constructed in Theorem 3.2 will add to the measure ρ so that the potential $\mathbf{u}$ will be a Kantorovich potential for the classical transport problem on $\overline{\Omega}$ between $\mu := \rho^+\mathcal{L}^N \llcorner \Omega + \theta^+$ and $\nu := \rho^-\mathcal{L}^N \llcorner \Omega + \theta^-$, that is,

$$\int_{\overline{\Omega}} \mathbf{u}\mathrm{d}(\mu - \nu) = \min_{\gamma\in\Gamma(\mu,\nu)} \int_{\overline{\Omega}\times\overline{\Omega}} d_H(x,y)\mathrm{d}\gamma(x,y),$$

where $\Gamma(\mu,\nu) := \{\gamma \in \mathcal{M}^+(\overline{\Omega}\times\overline{\Omega}) : \ (\pi_x)_\sharp\gamma = \mu, (\pi_y)_\sharp\gamma = \nu\}$ denotes the set of transport plans from μ to ν on $\overline{\Omega}$.

PROPOSITION 4.1. *Let* $\mathbf{u}$ *be the limit of the family of Finsler p-Laplace problems constructed in Theorem* 3.2. *Then* $\mathbf{u}$ *is a Kantorovich potential for the classical optimal transport problem between* $\rho^+\mathcal{L}^N \llcorner \Omega + \theta^+$ *and* $\rho^-\mathcal{L}^N \llcorner \Omega + \theta^-$. *Moreover*

$$\int_\Omega \mathbf{u}\rho\mathrm{d}x = \min(\mathcal{K})_H.$$

Proof. In the definition of $\Theta_p \cdot \mathbf{n}$ in (2.17), we take as a test function $\eta = \mathbf{u}$ to get

$$\int_{\partial\Omega} \mathbf{u} \mathrm{d}(\Theta_p \cdot \mathbf{n}) = \int_\Omega \Theta_p \cdot \nabla \mathbf{u} \mathrm{d}x - \int_\Omega \mathbf{u}\rho \mathrm{d}x.$$

Thanks to Theorem 3.2, passing to the limit $p \to \infty$ (up to a subsequence) we get

$$\lim_{p\to\infty} \int_\Omega \Theta_p \cdot \nabla \mathbf{u} \mathrm{d}x = \int_{\partial\Omega} \mathbf{u} \mathrm{d}\theta + \int_\Omega \mathbf{u}\rho \mathrm{d}x. \tag{4.2}$$

Since $\mathbf{u}$ is $1-$Lipschitz with respect to d_H, we may find, thanks to Lemma 5.2, a sequence of smooth functions w_ϵ converging uniformly to $\mathbf{u}$ and enjoying the property of being $1-$Lipschitz with respect to d_H. By definition of $\Theta_p \cdot \mathbf{n}$, we get

$$\int_{\partial\Omega} (\mathbf{u} - w_\epsilon) \mathrm{d}(\Theta_p \cdot \mathbf{n}) = \int_\Omega \Theta_p \cdot (\nabla \mathbf{u} - \nabla w_\epsilon) \mathrm{d}x - \int_\Omega (\mathbf{u} - w_\epsilon)\rho \mathrm{d}x.$$

Taking $p \to \infty$ (again, up to a subsequence) and keeping in mind (4.2), we get

$$\int_\Omega \mathbf{u}\rho \mathrm{d}x + \int_{\partial\Omega} \mathbf{u} \mathrm{d}\theta = \int_\Omega (\mathbf{u} - w_\epsilon)\rho \mathrm{d}x + \int_{\partial\Omega} (\mathbf{u} - w_\epsilon) \mathrm{d}\theta + \int_\Omega \Theta \cdot \nabla w_\epsilon \mathrm{d}x = A_\epsilon + B_\epsilon, \tag{4.3}$$

with $A_\epsilon = \int_\Omega (\mathbf{u} - w_\epsilon)\rho \mathrm{d}x + \int_{\partial\Omega} (\mathbf{u} - w_\epsilon) \mathrm{d}\theta$ and $B_\epsilon = \int_\Omega \Theta \cdot \nabla w_\epsilon \mathrm{d}x$. Since w_ϵ converges uniformly to $\mathbf{u}$ on $\overline{\Omega}$, we have that $A_\epsilon \to 0$ as $\epsilon \to 0$. We claim that

$$B_\epsilon \to \int_\Omega H\left(x, \frac{\Theta}{|\Theta|}\right) \mathrm{d}|\Theta|$$

as $\epsilon \to 0$. We first observe that

$$\begin{aligned}
\int_\Omega \mathbf{u}\rho \mathrm{d}x &= \lim_{\epsilon\to 0} \int_\Omega w_\epsilon \rho \mathrm{d}x \\
&\le \lim_{\epsilon\to 0} \int_\Omega \nabla w_\epsilon \frac{\Theta}{|\Theta|} \mathrm{d}|\Theta| + \int_{\partial\Omega} \psi \mathrm{d}\theta^- - \int_{\partial\Omega} \phi \mathrm{d}\theta^+ \\
&\le \lim_{\epsilon\to 0} \int_\Omega H^*(x, \nabla w_\epsilon) H\left(x, \frac{\Theta}{|\Theta|}\right) \mathrm{d}|\Theta| + \int_{\partial\Omega} \psi \mathrm{d}\theta^- - \int_{\partial\Omega} \phi \mathrm{d}\theta^+ \\
&\le \int_\Omega H\left(x, \frac{\Theta}{|\Theta|}\right) \mathrm{d}|\Theta| + \int_{\partial\Omega} \psi \mathrm{d}\theta^- - \int_{\partial\Omega} \phi \mathrm{d}\theta^+,
\end{aligned}$$

where we have used Lemma 5.2 for the last inequality.

Again we proceed as in the proof of Theorem 3.2: since $\Theta_p \rightharpoonup \Theta$, we have by Reshetnyak's lower semicontinuity theorem that we get

$$\begin{aligned}
\int_\Omega H\left(x, \frac{\Theta}{|\Theta|}\right) \mathrm{d}|\Theta| &\le \liminf_p \int_\Omega H\left(x, \frac{\Theta_p}{|\Theta_p|}\right) \mathrm{d}|\Theta_p| \\
&= \liminf_p \int_\Omega H\Big(x, H^*(x, \nabla u_p)^{p-1} \partial_\xi H^*(x, \nabla u_p)\Big) \mathrm{d}x \\
&= \liminf_p \int_\Omega H^*(x, \nabla u_p)^{p-1} H(x, \partial_\xi H^*(x, \nabla u_p)) \mathrm{d}x
\end{aligned}$$

$$\le \liminf_p \left(\int_\Omega H^*(x, \nabla u_p)^p \mathrm{d}x \right)^{\frac{p-1}{p}}$$
$$= \liminf_p \left(\int_\Omega H^*(x, \nabla u_p)^{p-1}\ \partial_\xi H^*(x, \nabla u_p) \cdot \nabla u_p \mathrm{d}x \right)^{\frac{p-1}{p}}$$
$$= \liminf_p \left(\int_\Omega \nabla u_p \mathrm{d}\Theta_p \right)^{\frac{p-1}{p}}$$
$$= \int_\Omega \mathbf{u}\rho \mathrm{d}x + \int_{\partial\Omega} \mathbf{u} \mathrm{d}\theta$$
$$= \lim_{\epsilon\to 0} \int_\Omega w_\epsilon \rho \mathrm{d}x + \int_{\partial\Omega} w_\epsilon \mathrm{d}\theta,$$

where we have used Hölder's inequality combined with (2.8) and (2.10). Coming back to (4.3) we get

$$\int_\Omega \mathbf{u}\rho \mathrm{d}x + \int_{\partial\Omega} \mathbf{u}\mathrm{d}\theta = \int_\Omega H\left(x, \frac{\Theta}{|\Theta|}\right) \mathrm{d}|\Theta|.$$

To conclude, let us observe that, taking $v \in W^{1,\infty}(\Omega)$ such that $H^*(x, \nabla v(x)) \le 1$, we have

$$\int_\Omega \mathbf{u}\rho \mathrm{d}x + \int_{\partial\Omega} \mathbf{u}\mathrm{d}\theta = \int_\Omega H\left(x, \frac{\Theta}{|\Theta|}\right) \mathrm{d}|\Theta|$$
$$\ge \int_\Omega \frac{\Theta}{|\Theta|} \cdot \nabla v \mathrm{d}|\Theta|$$
$$= \int_\Omega \nabla v \mathrm{d}\Theta = \int_\Omega v\rho \mathrm{d}x + \int_{\partial\Omega} v \mathrm{d}\theta.$$

Thanks to (4.1) and the classical Kantorovich duality, we have

$$\int_\Omega \mathbf{u}\rho \mathrm{d}x + \int_{\partial\Omega} \mathbf{u}\mathrm{d}\theta = \int_{\overline{\Omega}\times\overline{\Omega}} d_H(x,y)\mathrm{d}\gamma(x,y),$$

where γ is an optimal plan of

$$\min\left\{ \int_{\overline{\Omega}\times\overline{\Omega}} d_H(x,y)\mathrm{d}\gamma(x,y) :\ (\pi_x)_\sharp\gamma = \rho^+\mathcal{L}^N \llcorner \Omega + \theta^+, (\pi_y)_\sharp\gamma = \rho^-\mathcal{L}^N \llcorner \Omega + \theta^- \right\}.$$

Since $(\pi_x)_\sharp\gamma \llcorner \partial\Omega = \theta^+$ and $(\pi_y)_\sharp\gamma \llcorner \partial\Omega = \theta^-$ we deduce that

$$\int_\Omega \mathbf{u}\rho \mathrm{d}x = \int_{\overline{\Omega}\times\overline{\Omega}} d_H(x,y)d\gamma(x,y) + \int_{\partial\Omega} \psi \mathrm{d}\theta^- - \int_{\partial\Omega} \phi \mathrm{d}\theta^+ = \min(\mathcal{K})_H.$$

□

5. Appendix. Let us recall some facts concerning the notion of tangential gradient which played an important role in the previous proofs. To give a glimpse on the necessity to introduce this notion, let us remember that Beckmann's transportation problem is an optimization problem on measure space under a divergence constraint. More particularly, the flow satisfies $-\operatorname{div}(\Phi) = \mu \in \mathcal{M}_b(\overline{\Omega})$. To do further analysis on such a problem and particularly to derive its dual problem we naturally attempt to integrate by parts in the divergence constraint and write, for some Lipschitz function u,

$$\int \nabla u \cdot \sigma \, \mathrm{d}\gamma = \int u \mathrm{d}\mu,$$

where $\gamma = |\Phi|$ and $\sigma = \frac{\Phi}{|\Phi|}$. Observe that ∇u may not be well defined on a $|\Phi|$-positive measure set, and thus the previous formula may not have sense. Thanks to [3] it is possible to give a sense to the previous formula using the notion of tangential gradient as follows. First we can define the tangent space to the measure γ

$$\mathcal{X}_\gamma(x) = \gamma - \mathrm{ess} \cup \Big\{ \sigma(x) : \ \sigma \in L^1_\gamma(\overline{\Omega}, \mathbb{R}^N), \ \mathrm{div}(\sigma\gamma) \in \mathcal{M}_b(\overline{\Omega}) \Big\}.$$

Then, the tangential gradient $\nabla_\gamma u(x)$ to a function $u \in C^1(\overline{\Omega})$ at x with respect to the measure γ is the orthogonal projection of $\nabla u(x)$ onto $\mathcal{X}_\gamma(x)$. Denoting by $\mathbf{P}_\gamma(x)$ the orthogonal projection on $\mathcal{X}_\gamma(x)$, it has been shown in [4] that the linear operator $u \in C^1(\overline{\Omega}) \to \nabla_\gamma u(x) := \mathbf{P}_\gamma(x)\nabla u(x) \in L^\infty_\gamma(\overline{\Omega}, \mathbb{R}^N)$ can be uniquely extended to a linear continuous operator

$$\nabla_\gamma : u \in \mathrm{Lip}(\overline{\Omega}) \to \nabla_\gamma u \in L^\infty_\gamma(\overline{\Omega}, \mathbb{R}^N).$$

Moreover, we have the following useful integration by parts formula

PROPOSITION 5.1 ([4]). *Given $\gamma \in \mathcal{M}^+_b(\overline{\Omega})$ and $\upsilon \in L^1_\gamma(\overline{\Omega}, \mathbb{R}^N)$ such that $\upsilon(x) \in \mathcal{X}_\gamma(x)$ for $\gamma-$a.e x and* $\mathrm{div}(\gamma\upsilon) := \rho \in \mathcal{M}_b(\overline{\Omega})$. *One then has*

$$\int_{\overline{\Omega}} u \mathrm{d}\rho = \int_{\overline{\Omega}} \upsilon \nabla_\gamma u \mathrm{d}\gamma$$

for any $u \in \mathrm{Lip}(\overline{\Omega})$.

To end this section let us recall the following useful approximation result [19, Lemma A.1] (see also [25, Lemma 3.1] for degenerate case of H).

LEMMA 5.2. *Let H be a nondegenerate Finsler metric and $u \in W^{1,\infty}(\Omega)$ such that $H^*(x, \nabla u(x)) \le 1$ for a.e. $x \in \Omega$. Then, there exists a sequence of $u_\epsilon \in C^1(\overline{\Omega})$ such that $u_\epsilon \rightrightarrows u$ uniformly on $\overline{\Omega}$ as $\epsilon \to 0$ and*

$$H^*(x, \nabla u_\epsilon(x)) \le 1 \textit{ for all } x \in \overline{\Omega}.$$

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